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LOCAL THETA CORRESPONDENCE AND MINIMAL K-TYPES OF POSITIVE DEPTH

BY

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ABSTRACT

In this paper we prove that two irreducible admissible representations of positive depth paired by the theta correspondence over a p-adic field have unrefined minimal K-types paired by the orbit correspondence. An application of our main result is that a positive depth character of a unitary group occurs as late as possible in the theta correspondence.

1. Introduction

Let $(U(\mathcal{V}), U(\mathcal{V}'))$ be a reductive dual pair of symplectic, orthogonal or unitary groups in a symplectic group $\mathrm{Sp}(\mathcal{W})$ over a p-adic field, i.e., $\mathcal{V}$ (resp. $\mathcal{V}'$) is a non-degenerate ϵ-hermitian (resp. ϵ'-hermitian) space over F or a quadratic extension of F, $\epsilon\epsilon' = -1$ and $\mathcal{W} = \mathcal{V} \odot \mathcal{V}'$. Let $\mathfrak{g}$ (resp. $\mathfrak{g}'$) be the Lie algebra of $U(\mathcal{V})$ (resp. $U(\mathcal{V}')$) and $\mathfrak{g}^*$ (resp. $\mathfrak{g}'^*$) be its dual space. Given a small admissible lattice chain $\mathcal{L}$ of period n in $\mathcal{V}$ (cf. section 2), we can define an open compact subgroup $G_{\mathcal{L},d/n}$ of $U(\mathcal{V})$ and lattices $\mathfrak{g}^*_{\mathcal{L},-d/n}$, $\mathfrak{g}^*_{\mathcal{L},(-d/n)^+}$ in $\mathfrak{g}^*$ for each non-negative integer d. Let $\widetilde{U(\mathcal{V})}$ denote the metaplectic cover of $U(\mathcal{V})$. We can regard $G_{\mathcal{L},d/n}$ as a subgroup via a splitting $\widetilde{\beta}$: $G_{\mathcal{L},d/n} \to \widetilde{G}_{\mathcal{L},d/n}$ with respect to a generalized lattice model of the Weil representation of the metaplectic cover $\widetilde{\mathrm{Sp}(\mathcal{W})}$ of the symplectic group $\mathrm{Sp}(\mathcal{W})$ (cf. [Pan01]). From [MP94], [MP96] and [Yu98b], we know that every irreducible admissible representation π of positive depth of $\widetilde{U(\mathcal{V})}$ has an unrefined minimal K-type of the form $(G_{\mathcal{L},d/n}, X + \mathfrak{g}^*_{\mathcal{L},(-d/n)^+})$ where X is a

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certain element in $\mathfrak{g}^*_{\mathcal{L},-d/n}$. Any two unrefined minimal K-types of an irreducible admissible representation have the same number d/n which is the depth of π.

It is known that there is a one-to-one correspondence between some irreducible admissible representations of $\widetilde{U(\mathcal{V})}$ and some irreducible admissible representations of $\widetilde{U(\mathcal{V}')}$ (cf. [Wal90]). This is the so-called **local theta correspondence** or **Howe dualty**. In [Pan02a], the author proves that the depths of two representations paired by the local theta correspondence are equal. In [Pan02b], the author proves that the depth zero minimal K-types of two representations paired by the local theta correspondence are paired by the theta correspondence for a certain finite reductive dual pair. In this paper we discuss the relation between minimal K-types of the two representations of positive depth paired by the local theta correspondence.

We have embeddings $\mathfrak{g} \to \mathfrak{sp}(\mathcal{W})$ and $\mathfrak{g}' \to \mathfrak{sp}(\mathcal{W})$. So we get $\mathfrak{sp}(\mathcal{W})^* \to \mathfrak{g}^*$ and $\mathfrak{sp}(\mathcal{W})^* \to \mathfrak{g}'^*$ by taking the dual maps. Embed $\mathcal{W}$ into $\mathfrak{sp}(\mathcal{W})^*$ as the union of $\{0\}$ and the minimal orbit. Then we get the moment maps $\mathfrak{M}$: $\mathcal{W} \to \mathfrak{g}^*$ and $\mathfrak{M}'$: $\mathcal{W} \to \mathfrak{g}'^*$ by restriction. The main result of this paper (Theorem 5.5) can be described as follows. Suppose that π (resp. π') is an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$ (resp. $\widetilde{U(\mathcal{V}')}$) such that π and π' correspond in the theta correspondence. Suppose that the depth of π is positive. Then there is an element w in $\mathcal{W}$ and a regular small admissible lattice chain $\mathcal{L}$ (resp. $\mathcal{L}'$) in $\mathcal{V}$ (resp. $\mathcal{V}'$) such that $\mathfrak{M}(w)+\mathfrak{g}^*_{\mathcal{L},(-d/n)^+}$ (resp. $\mathfrak{M}'(w)+\mathfrak{g}'^*_{\mathcal{L}',(-d/n)^+}$) is an unrefined minimal K-type of π (resp. π'). One immediate corollary (cf. Theorem 5.6) of this result is: if $(U(\mathcal{V}), U(\mathcal{V}'))$ is a reductive dual pair of unitary groups and π is a character of positive depth occurring in the theta correspondence, then the dimension of $\mathcal{V}'$ is greater than or equal to the dimension of $\mathcal{V}$. In other words, a character of positive depth of a unitary group does not occur early in the theta correspondence. This is an interesting phenomenon in the local theta correspondence. In fact, if we take the preservation principle conjecture for granted, we can determine when a character of positive depth will occur in the theta correspondence (cf. subsection 5.7). Further applications of our main result will appear in some other papers by the author.

This paper is a continuation of the papers [Pan02a], [Pan02b]. In particular, notations and results in [Pan02a] are used here to a great extent. The content of this paper is as follows. In section 2 we introduce the basic setting of reductive dual pairs and the Weil representation of the metaplectic cover of a symplectic group. In section 3 we summarize some results on minimal K-types from [Pan02a]. In section 4, we define the moment maps and discuss the relation of moment maps and Cayley transforms. We have our main result in section 5. In section 6, we prove Proposition 4.3. In section 7, we prove Proposition 5.2. We prove Proposition 5.3 in the last section.

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2. Reductive dual pairs and the Weil representation

In this section, we provide the general setting of this work. Material in this section is well-known and can be found in [How79], [MVW87].

2.1. NOTATION. Let F be a nonarchimedean local field, $\mathcal{O}_F$ the ring of integers of F, $\mathfrak{p}_F$ the prime ideal, ϖ_F a uniformizer of $\mathcal{O}_F$, $\mathbf{f}_F := \mathcal{O}_F/\mathfrak{p}_F$ the (finite) residue field and τ_F the identity automorphism of F. We assume that the characteristic of $\mathbf{f}_F$ is odd, and fix a non-trivial (additive) character ψ of F.

Let E be a quadratic extension of F, $\mathcal{O}_E$ the ring of integers of E, ϖ_E a uniformizer of $\mathcal{O}_E$, $\mathbf{f}_E$ the residue field of E and τ_E the nontrivial automorphism of E over F. We make the choice such that $\varpi_E = \varpi_F$ if E is an unramified extension, and $\tau_E(\varpi_E) = -\varpi_E$ if E is ramified. Let (D, ϖ, τ) be one of the triples (F, ϖ_F, τ_F) or (E, ϖ_E, τ_E). Let $\mathcal{O}$ be the ring of integers, $\mathfrak{p}$ the maximal ideal and $\mathbf{f}$ the residue field of D.

2.2. REDUCTIVE DUAL PAIRS. Let $\mathcal{V}$ be a finite-dimensional vector space over D and ϵ be 1 or -1. A map $\langle,\rangle\colon \mathcal{V}\times\mathcal{V}\to D$ is called an **ϵ-hermitian form** if the following conditions are satisfied:

$$(2.2.1)\qquad \begin{gathered}\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle;\quad \langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle;\\ \langle xa, yb\rangle = \tau(a)\langle x,y\rangle b;\quad \langle x,y\rangle = \epsilon\tau(\langle y,x\rangle)\end{gathered}$$

for any $x, y\in\mathcal{V}$ and $a, b\in D$. The form is **non-degenerate** if $\langle x,y\rangle = 0$ for all $y\in\mathcal{V}$ implies $x=0$. The pair $(\mathcal{V},\langle,\rangle)$ is called a (non-degenerate) **ϵ-hermitian space** when $\langle,\rangle$ is a non-degenerate ϵ-hermitian form on $\mathcal{V}$.

Let $(\mathcal{V},\langle,\rangle)$ (resp. $(\mathcal{V}',\langle,\rangle')$) be an ϵ-hermitian (resp. ϵ'-hermitian) space over D such that $\epsilon\epsilon' = -1$. Define $\mathcal{W} := \mathcal{V}\odot_D\mathcal{V}'$, which will be denoted by $\mathcal{V}\otimes\mathcal{V}'$ later for simplicity. Define a skew-symmetric F-bilinear form $\langle\!\langle,\rangle\!\rangle$ on $\mathcal{W}$ by

$$(2.2.2)\qquad \langle\!\langle,\rangle\!\rangle := \mathrm{Trd}_{D/F}(\langle,\rangle\odot\tau\circ\langle,\rangle')$$

where $\mathrm{Trd}_{D/F}$ denotes the reduced trace from D to F. The pair $(U(\mathcal{V}), U(\mathcal{V}'))$ is called a **(type I) reductive dual pair** in $\mathrm{Sp}(\mathcal{W})$ where $U(\mathcal{V})$ (resp. $U(\mathcal{V}')$) is the group of isometries of $(\mathcal{V}, \langle,\rangle)$ (resp. $(\mathcal{V}', \langle,\rangle')$).

2.3. Weil representation and the metaplectic cover. Let $(\mathcal{W}, \langle\langle,\rangle\rangle)$ be a symplectic space over F. Let $\mathrm{H}(\mathcal{W})$ be the **Heisenberg group** associated to the symplectic space $(\mathcal{W}, \langle\langle,\rangle\rangle)$. Let $(\rho_\psi, \mathcal{S})$ be the irreducible representation of $\mathrm{H}(\mathcal{W})$ with the non-trivial central character ψ obtained by the Stone–Von Neumann theorem. It is known that the symplectic group $\mathrm{Sp}(\mathcal{W})$ acts on $\mathrm{H}(\mathcal{W})$. Define the **metaplectic cover** $\widetilde{\mathrm{Sp}(\mathcal{W})}$ of $\mathrm{Sp}(\mathcal{W})$ to be the topological subgroup of $\mathrm{Sp}(\mathcal{W}) \times \mathrm{Aut}(\mathcal{S})$ consisting of the pairs (g, M) satisfying the condition

$$M \circ \rho_\psi(h) = \rho_\psi(g.h) \circ M \tag{2.3.1}$$

for $g \in \mathrm{Sp}(\mathcal{W})$, $M \in \mathrm{Aut}(\mathcal{S})$ and any $h \in \mathrm{H}(\mathcal{W})$. The metaplectic group $\widetilde{\mathrm{Sp}(\mathcal{W})}$ comes equipped with a representation ω_ψ on $\mathcal{S}$ given by

$$\omega_\psi(g, M) := M. \tag{2.3.2}$$

The representation $(\omega_\psi, \mathcal{S})$ of $\widetilde{\mathrm{Sp}(\mathcal{W})}$ or the projective representation $(M, \mathcal{S})$ of $\mathrm{Sp}(\mathcal{W})$ is called the **Weil representation** or the **oscillator representation.**

2.4. Generalized lattice model of the Weil representation. Let $\mathfrak{p}_F^{\lambda_F}$ be the conductor of the character ψ of F, i.e., λ_F is the smallest integer such that the restriction $\psi|_{\mathfrak{p}_F^{\lambda_F}}$ is trivial. Let A be a **good lattice** in $\mathcal{W}$, i.e., a lattice in $\mathcal{W}$ such that $A^*\varpi_F \subseteq A \subseteq A^*$ where

$$A^* := \{w \in \mathcal{W} \mid \langle\langle w, a\rangle\rangle \in \mathfrak{p}^{\lambda_F} \text{ for all } a \in A\}. \tag{2.4.1}$$

The notion of good lattice depends on the conductor of ψ. It is known that A^*/A is a symplectic space over $\mathbf{f}_F$. Let K_A denote the stabilizer of A in $\mathrm{Sp}(\mathcal{W})$ and K'_A the subgroup of elements $g \in K_A$ such that $(g-1).A^* \subseteq A$. We know that K'_A is a normal subgroup of K_A and K_A/K'_A is isomorphic to the finite symplectic group $\mathrm{Sp}(A^*/A)$. Let $\bar{\psi}$ be the character of $\mathbf{f}_F$ defined by $\bar{\psi}(\bar{t}) := \psi(t\varpi_F^{\lambda_F-1})$ where $\bar{t}$ denotes the image of t in $\mathcal{O}_F/\mathfrak{p}_F = \mathbf{f}_F$ for $t \in \mathcal{O}_F$. Let $(\bar{\omega}_{\bar{\psi}}, S)$ be the Weil representation of $\mathrm{Sp}(A^*/A)$, and $\widetilde{\omega}_\psi$ the representation of K_A inflated from $\bar{\omega}_{\bar{\psi}}$. Let $\bar{\rho}_{\bar{\psi}}$ be the irreducible representation of the Heisenberg group $\mathrm{H}(A^*/A)$ on the space S obtained by the Stone–Von Neumann theorem. This representation is characterized by the central character $\bar{\psi}$. We know that $A^* \times \mathfrak{p}_F^{\lambda_F-1}$ is a subgroup of $\mathrm{H}(\mathcal{W})$ and there is a homomorphism $A^* \times \mathfrak{p}_F^{\lambda_F-1} \to \mathrm{H}(A^*/A)$ by $(a, t\varpi_F^{\lambda_F-1}) := (\bar{a}, \bar{t})$ where $\bar{a}$ denotes the image of $a \in A^*$ in A^*/A. Let $\widetilde{\rho}_\psi$

denote the representation of $A^* \times \mathfrak{p}_F^{\lambda_F - 1}$ inflated from $\bar{\rho}_{\bar{\psi}}$. Let $\mathcal{S}(A)$ denote the space of locally constant, compactly supported maps $f\colon \mathcal{W} \to S$ such that

$$f(a+w) = \psi(\frac{1}{2}\langle\langle w, a\rangle\rangle)\widetilde{\rho}_{\psi}(a).(f(w)), \tag{2.4.2}$$

for any $w \in \mathcal{W}$ and $a \in A^*$ (regarded as a subset of $A^* \times \mathfrak{p}_F^{\lambda_F - 1}$). For $g \in \mathrm{Sp}(\mathcal{W})$, we define $M[g] \in \mathrm{Aut}(\mathcal{S}(A))$ by

$$(M[g].f)(w) := \int_{A^*} \psi(\frac{1}{2}\langle\langle a, w\rangle\rangle)\widetilde{\rho}_{\psi}(a^{-1}).(f(g^{-1}.(a+w)))da \tag{2.4.3}$$

where $g \in \mathrm{Sp}(\mathcal{W})$, $f \in \mathcal{S}(A)$, $w \in \mathcal{W}$ and da is a Haar measure on A^*. It is easy to check that $(g, M[g])$ belongs to $\widetilde{\mathrm{Sp}(\mathcal{W})}$. We can normalize the Haar measure da such that

$$(M[k].f)(w) = \widetilde{\omega}_{\psi}(k).(f(k^{-1}.w)) \tag{2.4.4}$$

for $k \in K_A$, $f \in \mathcal{S}(A)$ and $w \in \mathcal{W}$. Let $\widetilde{K}_A$ be the inverse image of K_A in $\widetilde{\mathrm{Sp}(\mathcal{W})}$ under the extension $\widetilde{\mathrm{Sp}(\mathcal{W})} \to \mathrm{Sp}(\mathcal{W})$. The map $K_A \to \widetilde{K}_A$ given by $k \mapsto (k, M[k])$ defines a splitting of the extension $\widetilde{K}_A \to K_A$. Therefore, if we identify K_A as a subgroup of $\widetilde{\mathrm{Sp}(\mathcal{W})}$ by the splitting $k \mapsto (k, M[k])$, then $\omega_{\psi}(k)$ becomes $\omega_{\psi}(k, M[k]) = M[k]$. This model $(\omega_{\psi}, \mathcal{S}(A))$ is a **generalized lattice model** of the Weil representation of $\widetilde{\mathrm{Sp}(\mathcal{W})}$.

For a union of A^*-cosets Q in $\mathcal{W}$, we define

$$\mathcal{S}(A)_Q := \{f \in \mathcal{S}(A) \mid f \text{ has support in } Q\}. \tag{2.4.5}$$

For $w \in \mathcal{W}$, define $\mathcal{S}(A)_w := \mathcal{S}(A)_{w+A^*}$. The dimension of $\mathcal{S}(A)_w$ is clearly equal to the dimension of S. Suppose that B is a lattice in $\mathcal{W}$ such that $B^* \subseteq B$. It is known that the lattice B^* acts on the Weil representation $(\omega_{\psi}, \mathcal{S})$. Moreover, if $B^* \subseteq A \subseteq A^* \subseteq B$, then it is also known that $\mathcal{S}(A)^{B^*} = \mathcal{S}(A)_B$ (cf. Lemma 8.2 of [Pan02a]).

2.5. Local theta correspondence. From the definition of the form $\langle\langle , \rangle\rangle$ we know that there exists an embedding $\iota_{\mathcal{V}'}\colon U(\mathcal{V}) \to \mathrm{Sp}(\mathcal{W})$ depending on $\mathcal{V}'$. Let $\widetilde{U(\mathcal{V})}$ be the inverse image of $\iota_{\mathcal{V}'}(U(\mathcal{V}))$ in $\widetilde{\mathrm{Sp}(\mathcal{W})}$. Let $\widetilde{U(\mathcal{V}')}$ be defined similarly. One can check that $\widetilde{U(\mathcal{V})}$ and $\widetilde{U(\mathcal{V}')}$ centralize each other in $\widetilde{\mathrm{Sp}(\mathcal{W})}$. Let $(\omega_{\psi}, \mathcal{S})$ be the Weil representation of $\widetilde{\mathrm{Sp}(\mathcal{W})}$ with respect to the character ψ of F. Then $(\omega_{\psi}, \mathcal{S})$ can be regarded as a representation of $\widetilde{U(\mathcal{V})} \times \widetilde{U(\mathcal{V}')}$ via the restriction to $\widetilde{U(\mathcal{V})} \cdot \widetilde{U(\mathcal{V}')}$ and the homomorphism $\widetilde{U(\mathcal{V})} \times \widetilde{U(\mathcal{V}')} \to \widetilde{U(\mathcal{V})} \cdot \widetilde{U(\mathcal{V}')}$.

An irreducible admissible representation (π, V) of $\widetilde{U(\mathcal{V})}$ is said to **correspond to** an irreducible admissible representation (π', V') of $\widetilde{U(\mathcal{V}')}$ if there is a non-trivial $\widetilde{U(\mathcal{V})} \times \widetilde{U(\mathcal{V}')}$ map

$$\Pi\colon \mathcal{S} \longrightarrow V \odot_{\mathbf{C}} V'. \tag{2.5.1}$$

This establishes a correspondence, called the **local theta correspondence** between some irreducible admissible representations of $\widetilde{U(\mathcal{V})}$ and some irreducible admissible representations of $\widetilde{U(\mathcal{V}')}$. It is proved by R. Howe (cf. [MVW87] chapitre 5) and J.-L. Waldspurger (cf. [Wal90]) that the local theta correspondence is one-to-one (when the characteristic of $\mathbf{f}_F$ is odd).

3. Unrefined minimal K-types

Most of the definitions and results are from [Pan02a]. Details can be found there. Please also check [Yu98b] and [Yu98a].

3.1. LATTICE CHAINS. Let $(\mathcal{V}, \langle,\rangle)$ be a (finite-dimensional non-degenerate) ϵ-hermitian space over D. Let L be a lattice in $\mathcal{V}$, i.e., a (free) $\mathcal{O}$-module whose rank is equal to the dimension of $\mathcal{V}$. Fix an integer κ for $\mathcal{V}$. Define

$$L^* := \{v \in \mathcal{V} \mid \langle v, l\rangle \in \mathfrak{p}^\kappa \text{ for all } l \in L\}. \tag{3.1.1}$$

It is clear that L^* is also a lattice in $\mathcal{V}$. The lattice L^* is called the **dual lattice** (with respect to the integer κ) of L. The lattice L is called a **good lattice** if $L^*\varpi \subseteq L \subseteq L^*$. A good lattice is said to be **maximal** (resp. **minimal**) if it is a maximal element (resp. minimal element) in the set of all good lattices with partial order given by inclusion. Two lattices L_1, L_2 are said to be **similar** (notation: $L_1 \sim L_2$) if $L_1 = L_2\varpi^k$ for some integer k.

Recall that ψ is a character of F with conductoral exponent λ_F. Define $\lambda := \lambda_F$ if D is F or an unramified quadratic extension of F, $\lambda := 2\lambda_F - 1$ otherwise. Let κ (resp. κ') be the integer used to define the dual lattices in $\mathcal{V}$ (resp. $\mathcal{V}'$) in (3.1.1). We make the following assumption,

$$\kappa + \kappa' = \lambda, \tag{3.1.2}$$

throughout the paper. We also assume that the duality of lattices in $\mathcal{W}$ is defined with respect to the integer λ_F.

It is known that (cf. [HM89]) the affine building of $\mathrm{GL}(\mathcal{V})$ can be parameterized as collections of lattices in $\mathcal{V}$. Now we consider the analogue for other classical groups. A non-empty collection of lattices $\mathcal{L} := \{L_i\}_{i\in\mathbf{Z}}$ in $\mathcal{V}$ is called a **lattice chain** in $\mathcal{V}$ if the following conditions are satisfied:

(i) $\mathcal{L}$ is totally ordered by inclusion, i.e., $L_{i+1} \subseteq L_i$ for each i.

(ii) There exists a number n such that $L_{i+n} = L_i\varpi$ for all i.

(iii) Each lattice L_i is similar to a good lattice or the dual lattice of a good lattice.

The number n is called the **period** of $\mathcal{L}$. A lattice chain $\mathcal{L}$ is **regular** if $L_i \neq L_j$ whenever $i \neq j$. A lattice chain $\mathcal{L}$ is called a **small admissible lattice chain with numerical invariant** (n, n_0) where n is a positive integer and $n_0 = 0$ or 1 if it has period n and the following two conditions are also satisfied:

(iv) $L_i^* = L_{-i-n_0}$ for all i when n is even and $n_0 = 1$; $L_i^* = L_{-i-n_0}$ for all $i \not\equiv 0$ or $n/2 \pmod n$ when n is even and $n_0 = 0$; $L_i^* = L_{-i-n_0}$ for all $i \not\equiv (n-1)/2 \pmod n$ when n is odd and $n_0 = 1$; $L_i^* = L_{-i-n_0}$ for all $i \not\equiv 0 \pmod n$ when n is odd and $n_0 = 0$.

(v) $L^*_{\lfloor\frac{n-1-n_0}{2}\rfloor}\varpi \subseteq L_{\lfloor\frac{n-1-n_0}{2}\rfloor} \subseteq \cdots \subseteq L_1 \subseteq L_0 \subseteq L_0^*$ and $L^*_{-1} \subseteq L_{-1} \subseteq L_{-2} \subseteq \cdots \subseteq L_{-\lfloor\frac{n+n_0}{2}\rfloor} \subseteq L^*_{-\lfloor\frac{n+n_0}{2}\rfloor}\varpi^{-1} \subseteq L_{-\lfloor\frac{n+n_0}{2}\rfloor-1}$.

A lattice chain $\mathcal{L} := \{L_i\}_{i\in\mathbf{Z}}$ is **self-dual with numerical invariant** (n, n_0) if it has period n and $L_i^* = L_{-i-n_0}$ for all i. Clearly, a self-dual lattice chain is small admissible. It is also clear that a small admissible lattice chain for n even and $n_0 = 1$ is self-dual.

3.2. Open compact subgroups. Let $(G, G') := (U(\mathcal{V}), U(\mathcal{V}'))$ be a reductive dual pair. Let $\mathcal{L} := \{L_i\}_{i\in\mathbf{Z}}$ be a small admissible lattice chain in $\mathcal{V}$ with numerical invariant (n, n_0). Define

$$L_i^\sharp := L^*_{-i-n_0}. \tag{3.2.1}$$

Clearly we have $L_i^\sharp \subset L_{i-1}^\sharp$ and $L_{i+n}^\sharp = L_i^\sharp\varpi$ for any i. It is known that

$$L_i \subseteq L_i^\sharp \subseteq L_{i-1}$$

for any i (cf. Lemma 4.5 of [Pan02a]). For any positive integer d, we define

(3.2.2)
$$\begin{aligned} G_{\mathcal{L},d/n} &:= \{g \in G \mid (g-1).L_i \subseteq L_{i+d},\ (g-1).L_i^\sharp \subseteq L_{i+d}^\sharp \text{ for all } i\},\\ G_{\mathcal{L},(d/n)^+} &:= \{g \in G \mid (g-1).L_i \subseteq L_{i+d+1}^\sharp,\ (g-1).L_i^\sharp \subseteq L_{i+d} \text{ for all } i\}.\end{aligned}$$

It is known that $G_{\mathcal{L},d/n}$ and $G_{\mathcal{L},(d/n)^+}$ are open compact subgroups of $U(\mathcal{V})$ (cf. [MP94] and [Yu98b]). We know that $G_{\mathcal{L},(d/n)^+}$ is a normal subgroup of $G_{\mathcal{L},d/n}$ and the quotient group $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$ is abelian. If $\mathcal{L}$ is self-dual, then $L_i^\sharp = L_i$ for all i and (3.2.2) becomes

$$G_{\mathcal{L},d/n} = \{g \in G \mid (g-1).L_i \subseteq L_{i+d} \text{ for all } i\},$$
$$G_{\mathcal{L},(d/n)^+} = \{g \in G \mid (g-1).L_i \subseteq L_{i+d+1} \text{ for all } i\}.$$

Let $\mathfrak{g}$ be the Lie algebra of $U(\mathcal{V})$ and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Regard $\mathfrak{g}$ as a subspace of $\mathrm{End}_D(\mathcal{V})$ and define

$$(3.2.3)\qquad \begin{aligned} \mathfrak{g}_{\mathcal{L},d/n} &:= \{X \in \mathfrak{g} \mid X.L_i \subseteq L_{i+d},\ X.L_i^\sharp \subseteq L_{i+d}^\sharp \text{ for all } i\},\\ \mathfrak{g}_{\mathcal{L},(d/n)^+} &:= \{X \in \mathfrak{g} \mid X.L_i \subseteq L_{i+d+1}^\sharp,\ X.L_i^\sharp \subseteq L_{i+d} \text{ for all } i\} \end{aligned}$$

for any $d \in \mathbf{Z}$. Hence, $\{\mathfrak{g}_{\mathcal{L},d/n} \mid d \in \mathbf{Z}\}$ forms a filtration of lattices in $\mathfrak{g}$. We can define lattices $\mathfrak{g}^*_{\mathcal{L},d/n}$ and $\mathfrak{g}^*_{\mathcal{L},(d/n)^+}$ in $\mathfrak{g}^*$ similarly. Moreover, it is known that there is a natural isomorphism between the Pontrjagin dual $(G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+})^\wedge$ of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$ and the quotient $\mathfrak{g}^*_{\mathcal{L},-d/n}/\mathfrak{g}^*_{\mathcal{L},(-d/n)^+}$ of additive groups, i.e., a character of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$ can be regarded as a $\mathfrak{g}^*_{\mathcal{L},(-d/n)^+}$-coset in $\mathfrak{g}^*_{\mathcal{L},-d/n}$. Later on, we will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the trace form (cf. [MP94]). In particular, $\mathfrak{g}^*_{-d/n}$ will be identified with $\mathfrak{g}_{-d/n}$. Hence, a character of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$ can be regarded as a $\mathfrak{g}_{\mathcal{L},(-d/n)^+}$-coset in $\mathfrak{g}_{\mathcal{L},-d/n}$.

3.3. A Splitting. Let $\mathcal{L} := \{L_i \mid i \in \mathbf{Z}\}$ (resp. $\mathcal{L}' := \{L'_j \mid j \in \mathbf{Z}\}$) be a small admissible lattice chain in $\mathcal{V}$ (resp. $\mathcal{V}'$) with numerical invariant (n, n_0) (resp. (n, n'_0)). Fix an Iwahori subgroup I of $U(\mathcal{V})$. We know that each $G_{\mathcal{L},(d/n)^+}$ has a conjugate which is a subgroup of I. Moreover, we know that

$$gG_{\mathcal{L},(d/n)^+}g^{-1} = G_{g.\mathcal{L},(d/n)^+}$$

where $g.\mathcal{L} := \{g.L_i \mid L_i \in \mathcal{L}\}$. We shall only consider the lattice chains $\mathcal{L}$ such that $G_{\mathcal{L},(d/n)^+}$ is a subgroup of I.

For any integer s we define

$$(3.3.1)\qquad A_s(\mathcal{L},\mathcal{L}') := \bigcap_{i+j=s} L_i^\sharp \otimes L'_j + \bigcap_{i+j=s} L_i \otimes L'^\sharp_j.$$

Define $\nu := \lceil \frac{-n-n_0-n'_0+2}{2} \rceil$ and

$$(3.3.2)\qquad A := \begin{cases} L_0 \otimes L'_0, & \text{if } n=1,\ \mathcal{L} \text{ is self-dual and } n_0 = n'_0 = 0;\\ L_{-1} \otimes L'_{-1}\varpi, & \text{if } n=1,\ \mathcal{L} \text{ is self-dual and } n_0 = n'_0 = 1;\\ A_\nu(\mathcal{L},\mathcal{L}'), & \text{if either } n \geq 2 \text{ or } n=1,\ n_0+n'_0 = 1. \end{cases}$$

Then it is known that A is a good lattice in $\mathcal{W}$ (cf. Lemma 9.4 in [Pan02a]). Let K_A and K'_A be defined as in subsection 2.4.

LEMMA: *For any positive integer d, $\iota_{\mathcal{V}'}(G_{\mathcal{L},d/n})$ is a subgroup of K'_A where $\iota_{\mathcal{V}'}$ is the embedding given in subsection 2.5.*

Proof: Let g be an element in $\iota_{\mathcal{V}'}(G_{\mathcal{L},d/n})$. First we suppose that $n = 1$, $\mathcal{L}$ is self-dual and $n_0 = n'_0 = 0$. Now $A^* = L_0^* \otimes L_0'^* = L_0 \otimes L_0'^*$ and $G_{\mathcal{L},d/n} = \{g \in G \mid (g-1).L_0 \subseteq L_0\varpi^d\}$. Then we have $(g-1).A^* \subseteq L_0\varpi \otimes L_0'^* \subseteq A$.

Next suppose that $n = 1$, $\mathcal{L}$ is self-dual and $n_0 = n'_0 = 1$. We know that $L'^*_{-1} \subseteq L_{-1} \subseteq L'^*_{-1}\varpi^{-1}$ by (v) in subsection 2.1. Now $A^* = L^*_{-1} \otimes L'^*_{-1}\varpi^{-1} = L_{-1} \otimes L'^*_{-1}$ and $G_{\mathcal{L},d/n} = \{g \in G \mid (g-1).L_{-1} \subseteq L_{-1}\varpi^d\}$ from (3.2.2). Then we have $(g-1).A^* \subseteq L_{-1} \otimes L'^*_{-1}\varpi \subseteq A$.

Now we consider the remaining cases. It is not difficult to check that $A^* = \bigcap_{i+j=\rho} L_i^\sharp \otimes L'_j + \bigcap_{i+j=\rho} L_i \otimes L_j'^\sharp$ where $\rho := -\nu - n_0 - n'_0 - n + 1$. We have

$$(g-1).A^* \subseteq \bigcap_{i+j=\rho+d} L_i^\sharp \otimes L'_j + \bigcap_{i+j=\rho+d} L_i \otimes L_j'^\sharp.$$

Since $\nu := \lceil \frac{-n-n_0-n'_0+2}{2} \rceil$, it is clear that $\nu \leq \rho + d$ for any positive integer d. Hence, $(g-1).A^* \subseteq A$. So $\iota_{\mathcal{V}'}(G_{\mathcal{L},d/n})$ is a subgroup of K'_A. ∎

We know that the generalized lattice model $(\omega_\psi, \mathcal{S}(A))$ of the Weil representation gives a nice splitting $\widetilde{\beta}_A\colon K_A \to \widetilde{K}_A$ (cf. [Pan01]). Since $\iota_{\mathcal{V}'}(G_{\mathcal{L},d/n})$ is a subgroup of K_A, we will identify $G_{\mathcal{L},d/n}$ with $\widetilde{\beta}_A(\iota_{\mathcal{V}'}(G_{\mathcal{L},d/n}))$ for simplicity.

3.4. UNREFINED MINIMAL K-TYPES. Let (π, V) be an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$. Let d be a positive integer. A pair $(G_{\mathcal{L},d/n}, \zeta)$, where $\mathcal{L}$ is a regular small admissible lattice chain in $\mathcal{V}$ and ζ is a character of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$, is called an **unrefined minimal K-type** (of positive depth) of π if the following two conditions are satisfied.

(i) The fixed point set $V^{G_{\mathcal{L},(d/n)^+}}$ is non-trivial and the representation of the group $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$ on the space $V^{G_{\mathcal{L},(d/n)^+}}$ contains ζ.

(ii) The character ζ, realized as a $\mathfrak{g}^*_{\mathcal{L},(-d/n)^+}$-coset in $\mathfrak{g}^*_{\mathcal{L},-d/n}$, contains no nilpotent element.

The following proposition is from the results in [MP94], [MP96] and [Yu98b].

PROPOSITION: *Every irreducible admissible representation (π, V) of $\widetilde{U(\mathcal{V})}$ of positive depth has an unrefined minimal K-type $(G_{\mathcal{L},d/n}, \zeta)$ for some positive integer d and some regular small admissible lattice chain $\mathcal{L}$ of period n.*

3.5. Let $\mathcal{L} := \{L_i \mid i \in \mathbf{Z}\}$ be a regular small admissible lattice chain in $\mathcal{V}$ with numerical invariant (n, n_0). If d is a positive integer and $\mathcal{L}' := \{L'_i \mid i \in \mathbf{Z}\}$

is a small admissible lattice chain with numerical invariant (n, n_0') such that $-n - n_0 - n_0' - d$ is even, we define

$$B(\mathcal{L}, \mathcal{L}', d/n) := \bigcap_{i+j=\mu} L_i \otimes L_j'^{\sharp} \cap \bigcap_{i+j=\mu} L_i^{\sharp} \otimes L_j' \tag{3.5.1}$$

where $\mu := (-n - n_0 - n_0' - d)/2$. It is known that $B(\mathcal{L}, \mathcal{L}', d/n)^* \subseteq B(\mathcal{L}, \mathcal{L}', d/n)$ for any positive integer d where $B(\mathcal{L}, \mathcal{L}', d/n)^*$ is the dual lattice of $B(\mathcal{L}, \mathcal{L}', d/n)$ in $\mathcal{W}$. Therefore, from subsection 2.4 we know that the lattice $B(\mathcal{L}, \mathcal{L}', d/n)^*$ acts on the Weil representation $(\omega_\psi, \mathcal{S})$. For a given $\mathcal{L}$, let $\mathcal{Q}(d)$ denote the set of small admissible lattice chains in $\mathcal{V}'$ with numerical invariant (n, n_0') such that $n + n_0 + n_0' + d$ is even. The following proposition is from [Pan02a] proposition 6.3.

Proposition: *Let $(U(\mathcal{V}), U(\mathcal{V}'))$ be a reductive dual pair in $\mathrm{Sp}(\mathcal{W})$. Let d be a positive integer and $\mathcal{L}$ a regular small admissible lattice chain in $\mathcal{V}$ with numerical invariant (n, n_0). Let $(\omega_\psi, \mathcal{S})$ be a model of the Weil representation of $\widetilde{\mathrm{Sp}(\mathcal{W})}$. Then*

$$\mathcal{S}^{G_{\mathcal{L},(d/n)+}} = \omega_\psi(\mathcal{H}').\Big(\sum_{\mathcal{L}' \in \mathcal{Q}(d)} \mathcal{S}^{B(\mathcal{L},\mathcal{L}',d/n)^*}\Big) \tag{3.5.2}$$

where $\mathcal{H}'$ is the Hecke algebra of $\widetilde{U(\mathcal{V}')}$.

4. Moment maps

As usual, let $(G, G') := (U(\mathcal{V}), U(\mathcal{V}'))$ be a reductive dual pair and ψ a character of F of conductor $\mathfrak{p}_F^{\lambda_F}$.

4.1. Cayley transforms. The Lie algebra $\mathfrak{g}$ of $U(\mathcal{V})$ consists of elements $c \in \mathrm{End}_D(\mathcal{V})$ such that $\langle c.v, v'\rangle + \langle v, c.v'\rangle = 0$ for all $v, v' \in \mathcal{V}$. If c is an element of $\mathfrak{g}$ and $1 + c$ is invertible, then we denote the element $(1 - c)(1 + c)^{-1}$ by $u(c)$. Similarly, if u is an element of $U(\mathcal{V})$ and $1 + u$ is invertible, then we denote the element $(1 - u)(1 + u)^{-1}$ by $c(u)$. It is easy to check that $u(c)$ (resp. $c(u)$) belongs to $U(\mathcal{V})$ (resp. $\mathfrak{g}$) when it is defined. For $x, y \in \mathcal{V}$ we define an element $c_{x,y} \in \mathrm{End}_D(\mathcal{V})$ by

$$c_{x,y}.v = x\langle y, v\rangle - \epsilon y\langle x, v\rangle \tag{4.1.1}$$

where $v \in \mathcal{V}$. It is easy to check that $c_{x,y}$ belongs to $\mathfrak{g}$. Moreover, we know that every element in $\mathfrak{g}$ is an F-linear combination of elements of the form $c_{x,y}$ for

$x, y \in \mathcal{V}$. Define

$$u_{x,y} := u(c_{x,y}) \tag{4.1.2}$$

when $1 + c_{x,y}$ is invertible. If $u_{x,y}$ is defined for some $x, y \in \mathcal{V}$, then it is easy to check that $1 + u_{x,y}$ is invertible and $c(u_{x,y}) = c_{x,y}$.

4.2. THE TRACE FORM. Let $\mathfrak{B}$ denote the trace form of the Lie algebra $\mathfrak{g}$, i.e., $\mathfrak{B}\colon \mathfrak{g} \times \mathfrak{g} \to F$ is defined by

$$\mathfrak{B}(X, Y) := \mathrm{Trd}_{D/F}(\mathrm{tr}(XY)),$$

where $\mathrm{tr}\colon \mathrm{End}_D(\mathcal{V}) \to D$ denotes the usual trace map of matrices and $\mathrm{Trd}\colon D \to F$ is the reduced trace of D over F. The trace form $\mathfrak{B}'$ on $\mathfrak{g}'$ is defined similarly. We know that $\mathfrak{B}$ is a symmetric F-bilinear form on $\mathfrak{g}$. It is easy to check that

$$\begin{aligned} &c_{x,y} \circ c_{z,v}(a) \\ =&c_{x,y}(z\langle v, a\rangle - \epsilon v\langle z, a\rangle) \\ =&x(\langle y, z\rangle\langle v, a\rangle - \epsilon\langle y, v\rangle\langle z, a\rangle) - \epsilon y(\langle x, z\rangle\langle v, a\rangle - \epsilon\langle x, v\rangle\langle z, a\rangle) \\ =&x(\langle y, z\rangle\langle v, a\rangle - \epsilon\langle y, v\rangle\langle z, a\rangle) + y(\langle x, v\rangle\langle z, a\rangle - \epsilon\langle x, z\rangle\langle v, a\rangle) \end{aligned} \tag{4.2.1}$$

for any $x, y, z, v, a \in \mathcal{V}$. Let $e_1, \ldots, e_m$ be a D-basis of $\mathcal{V}$. Define $(e_i, e_j) := \delta_{ij}$ and extend the definition by F-linearity so that $(,)$ becomes a hermitian (or symmetric) form on $\mathcal{V}$, i.e., $(e_i a, e_j b) = \tau(a) b \delta_{ij}$. Then $\{e_1, \ldots, e_m\}$ becomes an orthonormal basis of $\mathcal{V}$ with respect to the form $(,)$. Hence, any element $x \in \mathcal{V}$ is written uniquely as $\sum_i e_i(x, e_i)$. We have $\mathrm{tr}(c_{x,y} \circ c_{z,v}) = \sum_i (c_{x,y} \circ c_{z,v}(e_i), e_i)$. Now from (4.2.1) we have

$$\begin{aligned} &\sum_i (c_{x,y} \circ c_{z,v}(e_i), e_i) \\ =&\sum_i (x(\langle y, z\rangle\langle v, e_i\rangle - \epsilon\langle y, v\rangle\langle z, e_i\rangle) + y(\langle x, v\rangle\langle z, e_i\rangle - \epsilon\langle x, z\rangle\langle v, e_i\rangle), e_i) \\ =&\sum_i (x, e_i)(\langle y, z\rangle\langle v, e_i\rangle - \epsilon\langle y, v\rangle\langle z, e_i\rangle) + (y, e_i)(\langle x, v\rangle\langle z, e_i\rangle - \epsilon\langle x, z\rangle\langle v, e_i\rangle). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathfrak{B}(c_{x,y}, c_{z,v}) &= \mathrm{Trd}_{D/F}(\mathrm{tr}(c_{x,y} \circ c_{z,v})) \\ &= \mathrm{Trd}_{D/F}\Big(\sum_i (x, e_i)(\langle y, z\rangle\langle v, e_i\rangle - \epsilon\langle y, v\rangle\langle z, e_i\rangle) \\ &\qquad + (y, e_i)(\langle x, v\rangle\langle z, e_i\rangle - \epsilon\langle x, z\rangle\langle v, e_i\rangle)\Big) \end{aligned}$$

$$= \operatorname{Trd}_{D/F}\Big(\sum_i \langle y,z\rangle\langle v,e_i(x,e_i)\rangle - \epsilon\langle y,v\rangle\langle z,e_i(x,e_i)\rangle + \langle x,v\rangle\langle z,e_i(y,e_i)\rangle - \epsilon\langle x,z\rangle\langle v,e_i(y,e_i)\rangle\Big).$$

Because $x = \sum_i e_i(x,e_i)$ and $y = \sum_i e_i(y,e_i)$, we have

$$\begin{aligned}
&\mathfrak{B}(c_{x,y}, c_{z,v})\\
&= \operatorname{Trd}_{D/F}(\langle x,v\rangle\langle z,y\rangle + \langle y,z\rangle\langle v,x\rangle - \epsilon(\langle x,z\rangle\langle v,y\rangle + \langle y,v\rangle\langle z,x\rangle))\\
&= \operatorname{Trd}_{D/F}(\langle z\tau(\langle x,v\rangle) - \epsilon v\tau(\langle x,z\rangle), y\rangle + \langle v\tau(\langle y,z\rangle) - \epsilon z\tau(\langle y,v\rangle), x\rangle)\\
&= \operatorname{Trd}_{D/F}(\langle \epsilon z\langle v,x\rangle - v\langle z,x\rangle), y\rangle + \langle \epsilon v\langle z,y\rangle - z\langle v,y\rangle, x\rangle)\\
&= \operatorname{Trd}_{D/F}(-\langle c_{z,v}.y, x\rangle + \epsilon\langle c_{z,v}.x, y\rangle)
\end{aligned}$$

where τ is the fixed involution of D over F given in subsection 2.1. Because any element X in $\mathfrak{g}$ is an F-linear combination of elements of the form $c_{z,v}$ and $\mathfrak{B}$ is F-bilinear, we have the following identity:

$$\mathfrak{B}(c_{x,y}, X) = \operatorname{Trd}_{D/F}(-\langle X.y, x\rangle + \epsilon\langle X.x, y\rangle) \tag{4.2.2}$$

for any $x, y \in \mathcal{V}$, $X \in \mathfrak{g} \subset \operatorname{End}_D(\mathcal{V})$.

4.3. Moment maps. Recall that $\mathcal{W} = \mathcal{V} \otimes \mathcal{V}'$ is an F-space. Let $\phi\colon \mathcal{V}\otimes\mathcal{V}' \to \operatorname{Hom}_D(\mathcal{V},\mathcal{V}')$ be the isomorphism of F-spaces given by

$$\phi(v_1 \otimes v_2)(x) := v_2\langle v_1, x\rangle \tag{4.3.1}$$

for $v_1, x \in \mathcal{V}$, $v_2 \in \mathcal{V}'$. Similarly, let $\phi'\colon \mathcal{V}\otimes\mathcal{V}' \to \operatorname{Hom}_D(\mathcal{V}',\mathcal{V})$ be the isomorphism of F-spaces given by

$$\phi'(v_1 \otimes v_2)(y) := v_1\langle v_2, y\rangle' \tag{4.3.2}$$

for $v_1 \in \mathcal{V}$, $v_2, y \in \mathcal{V}'$. Define the moment map $\mathfrak{M}_0\colon \mathcal{W} \to \operatorname{End}_D(\mathcal{V})$ by

$$\mathfrak{M}_0(w) := \phi'(w) \circ \phi(w). \tag{4.3.3}$$

Suppose that $w = \sum_i a_i \otimes b_i$ is an element in $\mathcal{W}$ for $a_i \in \mathcal{V}$, $b_i \in \mathcal{V}'$. Then

$$\begin{aligned}
\mathfrak{M}_0(w)(x) = \phi'(w)\circ\phi(w)(x) &= \phi'(w)\Big(\sum_i b_i\langle a_i, x\rangle\Big)\\
&= \sum_{i,j} a_j\langle b_j, b_i\langle a_i, x\rangle\rangle'\\
&= \sum_{i,j}\Big(\frac{1}{2}a_j\langle b_j, b_i\rangle'\langle a_i, x\rangle + \frac{1}{2}a_i\langle b_i, b_j\rangle'\langle a_j, x\rangle\Big).
\end{aligned}$$

Then from (2.2.1), we have

$$\mathfrak{M}_0(w)(x) = \sum_{i,j} \Big(\frac{1}{2} a_j \langle a_i \tau(\langle b_j, b_i\rangle'), x\rangle - \frac{\epsilon}{2} a_i \langle a_j \langle b_j, b_i\rangle', x\rangle\Big).$$

Hence, we conclude

$$\mathfrak{M}_0(w) = -\frac{\epsilon}{2} \sum_{i,j} c_{a_i, a_j \langle b_j, b_i\rangle'}. \tag{4.3.4}$$

In particular, the image of the map $\mathfrak{M}_0$ is in $\mathfrak{g}$. Define the map $\mathfrak{M}\colon \mathcal{W} \to \mathfrak{g}$ by

$$\mathfrak{M}(w) := -\epsilon \varpi_F^{1-\lambda_F} \mathfrak{M}_0(w) = \frac{1}{2} \varpi_F^{1-\lambda_F} \sum_{i,j} c_{a_i, a_j \langle b_j, b_i\rangle'}. \tag{4.3.5}$$

Similarly, we define the moment map $\mathfrak{M}_0'\colon \mathcal{W} \to \mathfrak{g}'$ by $w \mapsto \phi(w) \circ \phi'(w)$. If $w = \sum_i a_i \odot b_i$ is an element in $\mathcal{W}$, then we have

$$\mathfrak{M}_0'(w) := -\frac{\epsilon'}{2} \sum_{i,j} c_{b_i, b_j \langle a_j, a_i\rangle}$$

by the same computation as above. Define the map $\mathfrak{M}'\colon \mathcal{W} \to \mathfrak{g}'$ by

$$\mathfrak{M}'(w) := -\epsilon' \varpi_F^{1-\lambda_F} \mathfrak{M}_0'(w) = \frac{1}{2} \varpi_F^{1-\lambda_F} \sum_{i,j} c_{b_i, b_j \langle a_j, a_i\rangle}.$$

PROPOSITION: *Let $\mathcal{L}$ be a regular small admissible lattice chain in $\mathcal{V}$ with numerical invariant (n, n_0) and $\mathcal{L}'$ a small admissible lattice chain in $\mathcal{V}'$ with numerical invariant (n, n_0'). Let d be a positive integer such that $n + n_0 + n_0' + d$ is even. Suppose that w is an element in $B(\mathcal{L}, \mathcal{L}', d/n)$. Then $\mathfrak{M}(w)$ is in $\mathfrak{g}_{\mathcal{L}, -d/n}$.*

The proof of this proposition is in section 6.

5. Moment maps and unrefined minimal K-types

In this section we describe the relation between the orbit correspondence and unrefined minimal K-types of the two representations of positive depths paired in theta correspondence for a reductive dual pair $(G, G') := (U(\mathcal{V}), U(\mathcal{V}'))$. Material about moment maps and orbit correspondence can be found in [Ada87] and [KKS78].

5.1. Let (π, V) (resp. (π', V')) be an irreducible admissible representation of the covering group $\widetilde{U(\mathcal{V})}$ (resp. $\widetilde{U(\mathcal{V}')}$) such that π and π' are paired by the local theta correspondence, i.e., there exists a non-trivial $\widetilde{U(\mathcal{V})} \times \widetilde{U(\mathcal{V}')}$ map $\Pi\colon \mathcal{S} \to V \odot_{\mathbb{C}} V'$ where $\mathcal{S}$ is the Weil representation.

LEMMA: *Let A be the lattice defined in (3.3.2). Suppose that d is a positive integer. Then:*

(i) *A^* is contained in $B(\mathcal{L}, \mathcal{L}', d/n)$;*

(ii) *$(g-1).B(\mathcal{L}, \mathcal{L}', d/n)$ is contained in A for any $g \in G_{\mathcal{L},(d/n)^+}$.*

Proof: Part (i) is Lemma 10.7 of [Pan02a]. Part (ii) follows easily from Lemma 10.8 of [Pan02a]. ∎

5.2. Suppose that the depth of (π, V) is positive. Then we can assume that the depth is d/n for some positive integers d and n. Moreover, we know that V has non-trivial $G_{\mathcal{L},(d/n)^+}$-fixed vectors for some regular small admissible lattice chain $\mathcal{L}$ in $\mathcal{V}$ of period n.

PROPOSITION: *Let (π, V) be an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$ of positive depth. Then (π, V) has a minimal K-type $(G_{\mathcal{L},d/n}, \zeta)$ where $\mathcal{L}$ is a regular admissible lattice chain such that one of the following three conditions is satisfied:*

(i) *$\mathcal{L}$ is self-dual, and $n = 1$ or 2;*

(ii) *$\mathcal{L}$ is self-dual, and $\gcd(n, d) = 2$;*

(iii) *$\gcd(n, d) = 1$.*

The proof of this proposition is in section 7.

5.3. Suppose that (π, V) has a non-zero vector fixed by $G_{\mathcal{L},(d/n)^+}$ for some regular small admissible lattice chain $\mathcal{L}$ in $\mathcal{V}$ with numerical invariant (n, n_0). By Proposition 3.5 we know that there is a small admissible lattice chain $\mathcal{L}'$ in $\mathcal{V}'$ with numerical invariant (n, n_0') such that $n+n_0+n_0'+d$ is even such that the map $\Pi\colon \mathcal{S}^{B(\mathcal{L},\mathcal{L}',d/n)^*} \to V \otimes_{\mathbf{C}} V'$ has non-trivial image. Let A be the lattice defined in (3.3.2). We know that $A^* \subseteq B(\mathcal{L}, \mathcal{L}', d/n)$ by (i) of Lemma 5.1. Therefore, we know that $\mathcal{S}(A)^{B(\mathcal{L},\mathcal{L}',d/n)^*} = \mathcal{S}(A)_{B(\mathcal{L},\mathcal{L}',d/n)}$ from Lemma 8.2 of [Pan02a] where $\mathcal{S}(A)$ is the generalized lattice model of the Weil representation with respect to the good lattice A. Since $\mathcal{S}(A)_{B(\mathcal{L},\mathcal{L}',d/n)} = \sum_{w\in B(\mathcal{L},\mathcal{L}',d/n)} \mathcal{S}(A)_w$, there is a vector $w \in B(\mathcal{L}, \mathcal{L}', d/n)$ and a non-zero element $f \in \mathcal{S}(A)_w$ such that $\Pi(f)$ is non-trivial.

From subsection 3.3 we know that $G_{\mathcal{L},d/n}$ is regarded as a subgroup of K_A'. Suppose that $f \in \mathcal{S}(A)_w$ and $g \in G_{\mathcal{L},d/n}$. Then from our choice of splitting we have

$$(\omega_\psi(g).f)(w) = (M[g].f)(w) = \widetilde{\omega}_\psi(g).(f(g^{-1}.w))$$

from (2.4.4). Because g is in K'_A, the action $\widetilde{\omega}_\psi(g)$ becomes trivial. Hence, we have

$$(\omega_\psi(g).f)(w) = f(g^{-1}.w) = f((g^{-1}-1).w + w).$$

Because $(g^{-1}-1).w$ is contained in A by (ii) of Lemma 5.1, we have

$$(\omega_\psi(g).f)(w) = \psi\Big(\frac{1}{2}\langle\!\langle w, (g^{-1}-1).w\rangle\!\rangle\Big) f(w) = \psi\Big(\frac{1}{2}\langle\!\langle (g-1).w, w\rangle\!\rangle\Big) f(w)$$

from (2.4.2). Because $G_{\mathcal{L},d/n}$ is a subgroup of K'_A, we know that the map $\psi_w\colon g \mapsto \psi(\frac{1}{2}\langle\!\langle (g-1).w, w\rangle\!\rangle)$ is a character of $G_{\mathcal{L},d/n}$ by Lemma 8.5 of [Pan02a]. Thus $G_{\mathcal{L},d/n}$ acts on the one-dimensional space spanned by $\Pi(f)$ via the character ψ_w. By (i) of Proposition 6.2 in [Pan02a] we know that $G_{\mathcal{L},(d/n)+}$ acts trivially on the space $\Pi(f)$. Hence, we can and will regard ψ_w as a character of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)+}$.

From the previous paragraph we know that the action of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)+}$ on $V^{G_{\mathcal{L},(d/n)+}}$ contains the character ψ_w. If the $\mathfrak{g}_{\mathcal{L},(-d/n)+}$-coset in $\mathfrak{g}_{\mathcal{L},-d/n}$ presenting ψ_w contains a nilpotent element, then from Theorem 5.2 of [MP94] we know that the depth of π is strictly less than d/n. This contradicts our assumption that the depth of π is d/n. Hence, $(G_{\mathcal{L},d/n}, \psi_w)$ is an unrefined minimal K-type of (π, V). Now we consider the other side. We have the following proposition whose proof is in section 8. It was J.-K. Yu who observed (ii) of this proposition should be true.

PROPOSITION: *Let $\mathcal{L}$, d, n, $\mathcal{L}'$, w be given as above.*

(i) *Suppose that* $\gcd(d,n)=1$. *Then $\mathcal{L}'$ is regular.*
(ii) *Suppose that* $\gcd(d,n)=2$ *and $\mathcal{L}'$ is not regular. Then there exists a regular small admissible lattice chain $\mathcal{M}$ (resp. $\mathcal{M}'$) in $\mathcal{V}$ (resp. $\mathcal{V}'$) with numerical invariant $(n/2, m_0)$ (resp. $(n/2, m'_0)$) for some m_0, m'_0 such that $n/2 + d/2 + m_0 + m'_0$ is even and w belongs to $B(\mathcal{M}, \mathcal{M}', \frac{d/2}{n/2})$.*

Renaming $\mathcal{M}'$ as $\mathcal{L}'$ ($\mathcal{M}$ as $\mathcal{L}$, $n/2$ as n, and $d/2$ as d) if necessary, we may say that $\mathcal{L}'$ is a regular small admissible lattice chain in $\mathcal{V}'$. Define the character ψ'_w of $G'_{\mathcal{L}',d/n}$ by

$$\psi'_w\colon g' \mapsto \psi\Big(\frac{1}{2}\langle\!\langle (g'-1).w, w\rangle\!\rangle\Big)$$

for $g' \in G'_{\mathcal{L}',d/n}$. We know that the restriction of ψ'_w to $G'_{\mathcal{L}',(d/n)+}$ is trivial because w is in $B(\mathcal{L}, \mathcal{L}', d/n)$. Hence, we regard ψ'_w as a character of the quotient group $G'_{\mathcal{L}',d/n}/G'_{\mathcal{L}',(d/n)+}$. By the same argument before Proposition 5.3 we know that the $\mathfrak{g}'_{\mathcal{L}',(-d/n)+}$-coset in $\mathfrak{g}'_{\mathcal{L}',-d/n}$ presenting the character ψ'_w cannot contain any nilpotent element. Otherwise, the depth of (π', V') is strictly less than d/n. But we know that the depth of (π, V) is equal to the depth of (π', V') by Theorem

6.6 in [Pan02a]. Thus we see that $(G'_{\mathcal{L}',d/n}, \psi'_w)$ is an unrefined minimal K-type of (π', V').

We summarize what we have obtained up to now. Suppose that (π, V) and (π', V') are two irreducible admissible representations paired in the theta correspondence and the depth of one representation is positive. Then there is a regular small admissible lattice chain $\mathcal{L}$ (resp. $\mathcal{L}'$) in $\mathcal{V}$ (resp. $\mathcal{V}'$) of period n and a element $w \in B(\mathcal{L}, \mathcal{L}', d/n)$ such that $(G_{\mathcal{L},d/n}, \psi_w)$ (resp. $(G'_{\mathcal{L}',d/n}, \psi'_w)$) is an unrefined minimal K-type of (π, V) (resp. (π', V')).

5.4. Let $\mathcal{L}$ and d be given as in subsection 5.2. The Cayley transform induces an isomorphism of abelian groups between $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$ and $\mathfrak{g}_{\mathcal{L},d/n}/\mathfrak{g}_{\mathcal{L},(d/n)^+}$ defined by

$$g + G_{\mathcal{L},(d/n)^+} \mapsto (1-g)(1+g)^{-1} + \mathfrak{g}_{\mathcal{L},(d/n)^+} \tag{5.4.1}$$

for $g \in G_{\mathcal{L},d/n}$. The trace form $\mathfrak{B}$ of $\mathfrak{g}$ gives an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$. Moreover, it induces an isomorphism from $\mathfrak{g}_{\mathcal{L},d/n}/\mathfrak{g}_{\mathcal{L},(d/n)^+}$ to $\mathfrak{g}^*_{\mathcal{L},d/n}/\mathfrak{g}^*_{\mathcal{L},(d/n)^+}$ defined by

$$X + \mathfrak{g}_{\mathcal{L},(d/n)^+} \mapsto (Y \mapsto \mathfrak{B}(X,Y)) + \mathfrak{g}^*_{\mathcal{L},(d/n)^+} \tag{5.4.2}$$

for $X \in \mathfrak{g}_{\mathcal{L},d/n}$ and $Y \in \mathfrak{g}$. We know that there is a non-degenerate pairing

$$\mathfrak{g}_{\mathcal{L},d/n}/\mathfrak{g}_{\mathcal{L},(d/n)^+} \times \mathfrak{g}^*_{\mathcal{L},-d/n}/\mathfrak{g}^*_{\mathcal{L},(-d/n)^+} \to \mathfrak{f}_F \tag{5.4.3}$$

given by

$$(X + \mathfrak{g}_{\mathcal{L},(d/n)^+}, Z + \mathfrak{g}^*_{\mathcal{L},(-d/n)^+}) \mapsto Z(X) \pmod{\mathfrak{p}_F}$$

for $X \in \mathfrak{g}_{\mathcal{L},d/n}$ and $Z \in \mathfrak{g}^*_{\mathcal{L},-d/n}$. Therefore, we have the isomorphism between $\mathfrak{g}^*_{\mathcal{L},-d/n}/\mathfrak{g}^*_{\mathcal{L},(-d/n)^+}$ and the Pontrjagin dual $(\mathfrak{g}_{\mathcal{L},d/n}/\mathfrak{g}_{\mathcal{L},(d/n)^+})^\wedge$ given by

$$Z + \mathfrak{g}^*_{\mathcal{L},(-d/n)^+} \mapsto (X + \mathfrak{g}_{\mathcal{L},(d/n)^+} \mapsto \psi(Z(X)\varpi_F^{\lambda_F - 1})) \tag{5.4.4}$$

for $X \in \mathfrak{g}_{\mathcal{L},d/n}$ and $Z \in \mathfrak{g}^*_{\mathcal{L},-d/n}$ By (5.4.1), (5.4.2) and (5.4.4) we obtain an isomorphism between the Pontrjagin dual $(G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+})^\wedge$ of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$ and the quotient $\mathfrak{g}_{\mathcal{L},-d/n}/\mathfrak{g}_{\mathcal{L},(-d/n)^+}$. Therefore, there is an element $X_w \in \mathfrak{g}_{\mathcal{L},-d/n}$ such that the coset $X_w + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ represents the character ψ_w of $G_{\mathcal{L},d/n}/G_{\mathcal{L},(d/n)^+}$. In fact, the condition

$$\psi_w(g) = \psi(\mathfrak{B}(X_w, (1-g)(1+g)^{-1})\varpi_F^{\lambda_F - 1}) \tag{5.4.5}$$

must be satisfied for ψ_w, X_w and any $g \in G_{\mathcal{L},d/n}$. Now

$$(1+g)^{-1} = 2^{-1}\Big(1 + \Big(\frac{g-1}{2}\Big)\Big)^{-1} = 2^{-1}\Big(1 + \sum_{i=1}^{\infty}\Big(\frac{1-g}{2}\Big)^i\Big).$$

Clearly $\mathfrak{B}(X, 2^{-1}\sum_{i=1}^{\infty}(\frac{1-g}{2})^i \varpi_F^{-\lambda_F+1})$ belongs to the kernel of ψ. Hence, (5.4.5) becomes

$$\psi_w(g) = \psi\Big(\mathfrak{B}\Big(X_w, \frac{1}{2}(1-g)\Big)\varpi_F^{\lambda_F-1}\Big) \tag{5.4.6}$$

for any $g \in G_{\mathcal{L},d/n}$.

5.5. Let w, X_w be as in subsection 5.4. Now we want to investigate the relation between w and X_w in more detail. Assume that $w = \sum_i \gamma_i \odot \eta_i$ is an element in the lattice $B(\mathcal{L}, \mathcal{L}', d/n)$ for some $\gamma_i \in \mathcal{V}$, $\eta_i \in \mathcal{V}'$. From the definition of the form $\langle\!\langle , \rangle\!\rangle$ in subsection 2.2 we have

$$\begin{aligned}
&\psi\Big(\frac{1}{2}\langle\!\langle (g-1).w, w\rangle\!\rangle\Big)\\
=&\psi\Big(\frac{1}{2}\Big\langle\!\!\Big\langle \sum_i (g-1).\gamma_i \odot \eta_i, \sum_i \gamma_i \odot \eta_i \Big\rangle\!\!\Big\rangle\Big)\\
=&\psi\Big(\frac{1}{2}\operatorname{Trd}_{D/F}\Big(\sum_{i,j}\langle (g-1).\gamma_i, \gamma_j\rangle \tau(\langle \eta_i, \eta_j\rangle')\Big)\Big)\\
=&\psi\Big(\frac{1}{4}\operatorname{Trd}_{D/F}\Big(\sum_{i,j}\langle (g-1).\gamma_i, \gamma_j\rangle \tau(\langle \eta_i, \eta_j\rangle') + \langle (g-1).\gamma_j, \gamma_i\rangle \tau(\langle \eta_j, \eta_i\rangle')\Big)\Big)
\end{aligned}$$

for $g \in G_{\mathcal{L},d/n}$. Because $\epsilon\epsilon' = -1$ and $\operatorname{Trd}_{D/F}(ab) = \operatorname{Trd}_{D/F}(ba)$ for any two elements a, b in D, we have

$$\begin{aligned}
&\psi\Big(\frac{1}{2}\langle\!\langle (g-1).w, w\rangle\!\rangle\Big)\\
=&\psi\Big(\frac{1}{4}\operatorname{Trd}_{D/F}\Big(\sum_{i,j}\langle (g-1).\gamma_i, -\epsilon\gamma_j\langle \eta_j, \eta_i\rangle'\rangle + \langle (g-1).\gamma_j\langle \eta_j, \eta_i\rangle', \gamma_i\rangle\Big)\Big)\\
=&\psi\Big(\mathfrak{B}\Big(\sum_{i,j}\frac{1}{2}c_{\gamma_i,\gamma_j\langle\eta_j,\eta_i\rangle'}, \frac{1}{2}(1-g)\Big)\Big)\\
=&\psi\Big(\mathfrak{B}\Big(\sum_{i,j}\frac{1}{2}\varpi_F^{1-\lambda_F}c_{\gamma_i,\gamma_j\langle\eta_j,\eta_i\rangle'}, \frac{1}{2}(1-g)\Big)\varpi_F^{\lambda_F-1}\Big)
\end{aligned}$$

by (4.2.2). Hence, by (5.4.6), X_w can be chosen to be $\frac{1}{2}\varpi_F^{1-\lambda_F}\sum_{i,j} c_{\gamma_i,\gamma_j\langle\eta_j,\eta_i\rangle'}$, which is just $\mathfrak{M}(w)$ defined in subsection 4.3. Similarly, if $\mathcal{L}'$ is a small admissible lattice chain and g' is an element in $G'_{\mathcal{L}',d/n}$, then we have

$$\begin{aligned}\psi\Big(\frac{1}{2}\langle\!\langle(g'-1).w,w\rangle\!\rangle\Big) =&\psi\Big(\frac{1}{2}\operatorname{Trd}_{D/F}\Big(\sum_{i,j}\langle\gamma_i,\gamma_j\rangle\tau(\langle(g'-1).\eta_i,\eta_j\rangle')\Big)\Big)\\ =&\psi\Big(\frac{1}{2}\operatorname{Trd}_{D/F}\Big(\sum_{i,j}\langle(g'-1).\eta_i,\eta_j\rangle'\tau(\langle\gamma_i,\gamma_j\rangle)\Big)\Big)\\ =&\psi\Big(\mathfrak{B}'\Big(\sum_{i,j}\frac{1}{2}c_{\eta_i,\eta_j\langle\gamma_j,\gamma_i\rangle_1},\frac{1}{2}(1-g')\Big)\Big)\\ =&\psi\Big(\mathfrak{B}'\Big(\sum_{i,j}\frac{1}{2}\varpi_F^{1-\lambda_F}c_{\eta_i,\eta_j\langle\gamma_j,\gamma_i\rangle_1},\frac{1}{2}(1-g')\Big)\varpi_F^{\lambda_F-1}\Big).\end{aligned}$$

The last equality is obtained by the same argument as above. Note that we have used the property $\operatorname{Trd}_{D/F}(a)=\operatorname{Trd}_{D/F}(\tau(a))$ for $a\in D$. We summarize our discussion as the following theorem which is our main result in this paper.

THEOREM: *Let $(G,G'):=(U(\mathcal{V}),U(\mathcal{V}'))$ be a reductive dual pair of orthogonal, symplectic or unitary groups. Let (π,V) (resp. (π',V')) be an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$ (resp. $\widetilde{U(\mathcal{V}')}$). Suppose that π,π' correspond in the theta correspondence and the depth of (π,V) is positive. Then there exist a positive integer d, an element $w\in\mathcal{W}$ and a regular small admissible lattice chain $\mathcal{L}$ (resp. $\mathcal{L}'$) in $\mathcal{V}$ (resp. $\mathcal{V}'$) of period n such that $(G_{\mathcal{L},d/n},\mathfrak{M}(w)+\mathfrak{g}_{\mathcal{L},(-d/n)^+})$ (resp. $(G'_{\mathcal{L}',d/n},\mathfrak{M}'(w)+\mathfrak{g}'_{\mathcal{L}',(-d/n)^+})$) is a minimal K-type of (π,V) (resp. (π',V')).*

5.6. AN APPLICATION. In this subsection we give an application of Theorem 5.5 to theta correspondence for a dual pair of unitary groups. Now we assume that D is a quadratic extension of F. Hence, both $U(\mathcal{V})$ and $U(\mathcal{V}')$ are unitary groups. Let χ be a non-trivial character of $\widetilde{U(\mathcal{V})}$ of positive depth. It is not difficult to see that the unrefined minimal K-types of χ must be of the form $(G_{\mathcal{L},d/n},\zeta)$ for some positive integers d and n where ζ is presented by a coset $X+\mathfrak{g}_{\mathcal{L},-(d/n)^+}$ and X is a non-zero scaler matrix in $\mathfrak{g}\subseteq\operatorname{End}(\mathcal{V})$. Suppose that σ is an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$ of depth strictly less than the depth of χ. Then it is clear that $\sigma\otimes\chi$ is an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$ of depth equal to the depth of χ. The coset $X+\mathfrak{g}_{\mathcal{L},-(d/n)^+}$ still presents the second component of an unrefined minimal K-type of $\sigma\otimes\chi$. Moreover, all unrefined minimal K-types of $\sigma\otimes\chi$ must be of this form.

THEOREM: *Let $(U(\mathcal{V}),U(\mathcal{V}'))$ be a reductive dual pair of two unitary groups. Suppose that σ is an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$ and χ is a* *character of $\widetilde{U(\mathcal{V})}$ of depth strictly greater than the depth of σ. If the representation $\sigma \odot \chi$ occurs in the theta correspondence, then the dimension of $\mathcal{V}'$ is greater than or equal to the dimension of $\mathcal{V}$.*

Proof: Let π' be the irreducible admissible representation of $\widetilde{U(\mathcal{V}')}$ corresponding to $\sigma \odot \chi$. By Theorem 5.5 we know that there exist an element $w \in \mathcal{W}$ and a regular small admissible lattice chain $\mathcal{L}$ (resp. $\mathcal{L}'$) in $\mathcal{V}$ (resp. $\mathcal{V}'$) such that $\mathfrak{M}(w) + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ (resp. $\mathfrak{M}'(w) + \mathfrak{g}'_{\mathcal{L}',(-d/n)^+}$) is the second component of an unrefined minimal K-type of $\sigma \otimes \chi$ (resp. π'). From the remark before the theorem we know that $\mathfrak{M}(w)$ must be a scalar matrix. In particular, the rank of the matrix $\mathfrak{M}(w)$ is equal to the dimension of $\mathcal{V}$. From (4.3.3) and (4.3.5) we see that the rank of $\mathfrak{M}(w)$ is less than or equal to the minimum of the dimensions of $\mathcal{V}$ and $\mathcal{V}'$. Hence, the dimension of $\mathcal{V}'$ has to be greater than or equal to the dimension of $\mathcal{V}$. ∎

This theorem says that certain irreducible admissible representations of $\widetilde{U(\mathcal{V})}$ cannot occur too early in the theta correspondence.

5.7.

COROLLARY: *Let $(U(\mathcal{V}), U(\mathcal{V}'))$ be a reductive dual pair of two unitary groups. Suppose that χ is a character of $\widetilde{U(\mathcal{V})}$ of positive depth and occurs in the theta correspondence. Then the dimension of $\mathcal{V}'$ is greater than or equal to the dimension of $\mathcal{V}$.*

Proof: This follows from the previous theorem by taking σ to be the trivial representation of $\widetilde{U(\mathcal{V})}$. ∎

It is well known that the corollary cannot be true if χ is not assumed to be of positive depth.

Let σ be an irreducible admissible representation and χ a character of $\widetilde{U(\mathcal{V})}$ such that the depth of σ is strictly less than the depth of χ. It is obvious that $\sigma \odot \chi$ is still an irreducible admissible representation of $\widetilde{U(\mathcal{V})}$. Let sgn denote the sign character of $U(\mathcal{V})$. We regard sgn as a character of $\widetilde{U(\mathcal{V})}$ via the extension $\widetilde{U(\mathcal{V})} \to U(\mathcal{V})$. Let $\mathcal{V}'^{\pm}$ be the ϵ'-hermitian space given by the condition

$$\epsilon_{E/F}((-1)^{\frac{m^{\pm}(m^{\pm}-1)}{2}} \det(\mathcal{V}'^{\pm})) = \pm 1$$

where $m^{\pm}$ is the dimension of $\mathcal{V}'^{\pm}$. Let m_0^+ (resp. m_0^-) denote the dimension of $\mathcal{V}'^{+}$ (resp. $\mathcal{V}'^{-}$) such that $\sigma \odot \chi$ (resp. $(\sigma \odot \chi) \odot \mathrm{sgn}$) first occurs in the theta correspondence for the reductive dual pair $(U(\mathcal{V}), U(\mathcal{V}'^{+}))$ (resp. $(U(\mathcal{V}), U(\mathcal{V}'^{-}))$).

It is conjectured in [MHS96] that

$$m_0^+ + m_0^- = 2l + 2 \tag{5.7.1}$$

where l is the dimension of $\mathcal{V}$. According to Theorem 5.6, we must have $m_0^+ \geq l$ and $m_0^- \geq l$. Therefore, if the preservation principle (5.7.1) is true, then we conclude that

$$\{m_0^+, m_0^-\} = \begin{cases} \{l, l+2\}, & \text{when } m_0^\pm \text{ and } l \text{ are of the same parity;} \\ \{l+1, l+1\}, & \text{when } m_0^\pm \text{ and } l \text{ are of the opposite parity.} \end{cases}$$

6. Proof of Proposition 4.3

6.1.

LEMMA: *Let $\mathcal{L}$ be a regular small admissible lattice chain of period n in $\mathcal{V}$, d a positive integer and κ the integer in (3.1.1). Then $\mathfrak{g}_{\mathcal{L},-d/n}$ is generated by those $c_{x,y}$'s with $x, y \in \mathcal{V}$ satisfying the following two conditions:*

(i) *x in $L_i^\sharp$, y in L_j for some i, j such that $i + j \geq -d - 1 + n - n_0 - \kappa n$;*
(ii) *x in $L_{i'}$, y in $L_{j'}^\sharp$ for some i', j' such that $i' + j' \geq -d - 1 + n - n_0 - \kappa n$.*

Proof: By Proposition 7.3 of [Pan02a] we know that $\mathfrak{g}_{\mathcal{L},-d/n}$ is generated by those $c_{x,y}$'s with $x, y \in \mathcal{V}$ satisfying the following conditions:

$$\begin{aligned} &\mathrm{ord}_{L_j^*}(x) + \mathrm{ord}_{L_{j+d}}(y) \geq -\kappa; \quad \mathrm{ord}_{L_j^*}(y) + \mathrm{ord}_{L_{j+d}}(x) \geq -\kappa; \\ &\mathrm{ord}_{(L_j^\sharp)^*}(x) + \mathrm{ord}_{L_{j+d}^\sharp}(y) \geq -\kappa; \quad \mathrm{ord}_{(L_j^\sharp)^*}(y) + \mathrm{ord}_{L_{j+d}^\sharp}(x) \geq -\kappa \end{aligned} \tag{6.1.1}$$

for all $j \in \mathbf{Z}$. By the same proof of Corollary 7.7 in [Pan02a] we can show that the conditions in (6.1.1) are equivalent to (i) and (ii) of the lemma. ∎

6.2.

LEMMA: *Let $\mathcal{L} = \{L_i\}_{i \in \mathbf{Z}}$ be a small admissible lattice chain with numerical invariant (n, n_0), and κ the number defining the duality of lattices in (3.1.1). Then*

$$\langle L_i, L_j^\sharp \rangle \subseteq \mathfrak{p}^{\kappa + \lfloor (i+j+n_0)/n \rfloor}, \tag{6.2.1}$$

for any integers i, j.

Proof: We have $L_j^\sharp = L^*_{-j-n_0}$ from (3.2.1). So if $-j - n_0 + n > i \geq -j - n_0$, then $\langle L_i, L_j^\sharp \rangle \subseteq \mathfrak{p}^\kappa = \mathfrak{p}^{\kappa + \lfloor (i+j+n_0)/n \rfloor}$. Since $L_{i+kn} = L_i \varpi^k$ for all i and k, the lemma follows easily. ∎

6.3. Proof of Proposition 4.3.

Proof: Suppose that w is an element in $B(\mathcal{L}, \mathcal{L}', d/n)$. Write $w = \sum_i a_i \odot b_i$ for $a_i \in \mathcal{V}$, $b_i \in \mathcal{V}'$. We know that

$$(6.3.1)\qquad \begin{aligned} B(\mathcal{L}, \mathcal{L}', d/n) &= \bigcap_{i+j=\mu} L_i \odot L_j'^{\sharp} \cap \bigcap_{i+j=\mu} L_i^{\sharp} \odot L_j' \\ &= \sum_{i+j=\mu+n-1} L_i \odot L_j'^{\sharp} \cap \sum_{i+j=\mu+n-1} L_i^{\sharp} \odot L_j' \end{aligned}$$

where $\mu := (-n - n_0 - n_0' - d)/2$. Suppose that $a_i \in L_k$, $b_i \in L'^{\sharp}_{k'}$, $a_j \in L^{\sharp}_l$, $b_j \in L'_{l'}$ for some k, k', l, l'. Then we can require that

$$(6.3.2)\qquad \begin{aligned} k + k' &\geq \mu + n - 1, \\ l + l' &\geq \mu + n - 1 \end{aligned}$$

by (6.3.1). Now we have $\langle b_j, b_i\rangle' \in \mathfrak{p}^{\kappa' + \lfloor (k'+l'+n_0')/n \rfloor}$ by Lemma 6.2. Then

$$(6.3.3)\qquad a_j\langle b_j, b_i\rangle' \in L_l^{\sharp}\varpi^{\kappa'+\lfloor (k'+l'+n_0')/n\rfloor} = L^{\sharp}_{l+(\kappa'+\lfloor (k'+l'+n_0')/n\rfloor)n}.$$

Note that

$$\lambda = \begin{cases} \lambda_F & \text{if } D \text{ is unramified over } F; \\ 2\lambda_F - 1 & \text{if } D \text{ is ramified over } F. \end{cases}$$

Hence, $\mathfrak{p}_F^{1-\lambda_F} = \mathfrak{p}^{1-\lambda}$. Now we have $\frac{1}{2}\varpi_F^{1-\lambda_F} c_{a_i, a_j\langle b_j, b_i\rangle'} = \frac{1}{2} c_{a_i\varpi_F^{1-\lambda_F}, a_j\langle b_j, b_i\rangle'}$, so

$$(6.3.4)\qquad a_i\varpi_F^{1-\lambda_F} \in L_k\mathfrak{p}_F^{1-\lambda_F} = L_k\mathfrak{p}^{1-\lambda} = L_{k+(1-\kappa-\kappa')n}.$$

Then we have

$$\begin{aligned} & k + (1-\kappa-\kappa')n + l + (\kappa' + \lfloor (k'+l'+n_0')/n \rfloor)n \\ \geq & k + (1-\kappa-\kappa')n + l + \kappa' n + k' + l' + n_0' + 1 - n \\ \geq & -d - 1 + n - n_0 - \kappa n \end{aligned}$$

from (6.3.2). By the same argument we can show that $a_i\varpi^{1-\lambda_F}$ is in $L_r^{\sharp}$, $a_j\langle b_j, b_i\rangle'$ is in L_r for some r, s such that $r + s \geq -d - 1 + n - n_0 - \kappa n$. Therefore, $\frac{1}{2}\varpi_F^{1-\lambda_F} c_{a_i, a_j\langle b_j, b_i\rangle'} \in \mathfrak{g}_{\mathcal{L}, -d/n}$ by Lemma 6.1. From (4.3.5), we have $\mathfrak{M}(w) = \frac{1}{2}\varpi_F^{1-\lambda_F}\sum_{i,j} c_{a_i, a_j\langle b_j, b_i\rangle'}$. Thus the proposition is proved. ∎

7. Proof of Proposition 5.2

Proposition 7.1 is used in the proof of Proposition 5.2 in subsection 7.2. In subsection 7.3–7.7 there are a few lemmas that will be used for the proof of Proposition 7.1 in subsection 7.8. Most of the approach in this section was suggested by Jiu-Kang Yu. Let G denote the classical group $U(\mathcal{V})$.

7.1.

PROPOSITION: *Let e be a non-negative integer and $\mathcal{M}$ be a regular small admissible lattice chain in $\mathcal{V}$ with numerical invariant (m, m_0). Then there exist a non-negative integer d and a regular small admissible lattice chain $\mathcal{L}$ with numerical invariant (n, n_0) satisfying the following conditions:*

(i) $d/n = e/m$;

(ii) $\gcd(d,n) = 1$; *or* $\gcd(d,n) = 2$ *and* $n_0 = 1$;

(iii) $G_{\mathcal{L},(d/n)^+} \subseteq G_{\mathcal{M},(e/m)^+}$.

The proof of this proposition is in subsection 7.8.

7.2. PROOF OF PROPOSITION 5.2.

Proof: Now we are ready to prove Proposition 5.2. By Proposition 3.4 we know that an irreducible admissible representation (π, V) of G of positive depth has a minimal K-type $(G_{\mathcal{M},e/m}, \xi)$ where $\mathcal{M}$ is a regular small admissible lattice chain of period m in $\mathcal{V}$.

First, suppose that $m = 1$ and $\mathcal{M} := \{M_i\}_{i\in\mathbf{Z}}$ is not self-dual. In this case, we know that $M_0 \neq M_0^*\varpi^k$ for any k and $M_0^* \notin \mathcal{M}$. Let $\mathcal{L}$ be the regular self-dual lattice chain generated by $\{M_0, M_0^*\}$. Clearly the numerical invariant of $\mathcal{L}$ is $(2,1)$. Moreover, it is easy to check that $G_{\mathcal{M},(e+1)/1} = G_{\mathcal{L},2(e+1)/2}$ and $G_{\mathcal{M},(e/1)^+} = G_{\mathcal{L},(2e/2)^+}$ for any non-negative integer e. Hence, it is clear that (π, V) has a minimal K-type of the form $(G_{\mathcal{L},2e/2}, \zeta)$ for some ζ. Clearly $\gcd(2e, 2) = 2$.

Next, suppose that $m > 1$. Because we assume that (π, V) has a minimal K-type $(G_{\mathcal{M},e/m}, \xi)$, we know that $V^{G_{\mathcal{M},(e/m)^+}}$ is non-trivial. By Proposition 7.1, we know that there is a regular small admissible lattice chain $\mathcal{L}$ with numerical invariant (n, n_0) and a positive integer d such that $G_{\mathcal{L},(d/n)^+} \subseteq G_{\mathcal{M},(e/m)^+}$. Hence, $V^{G_{\mathcal{L},(d/n)^+}}$ is also non-trivial. Therefore, (π, V) must have a minimal K-type of the form $(G_{\mathcal{L},d/n}, \zeta)$ for some ζ. Now we know that $\gcd(d,n) = 1$ or 2. If $\gcd(d,n) = 2$, then we also know that $n_0 = 1$. In this case n is even and it is known that $\mathcal{L}$ is self-dual from subsection 3.1. ∎

7.3. By proving Proposition 7.1. we need few lemmas.

LEMMA: *Let $\mathcal{M} := \{M_i\}_{i\in\mathbf{Z}}$ be a regular small admissible lattice chain of numerical invariant (m, m_0). If $M_t^* \in \mathcal{M}$ for some t, then $M_t^* = M_{-t-m_0}$.*

Proof: This is obvious from condition (iv) in subsection 3.1. ∎

7.4. Suppose that $\mathcal{M} := \{M_i\}_{i\in\mathbf{Z}}$ is a small admissible lattice chain of period kn with $k > 1$. Let t be an integer such that $0 \le t \le k-1$. Let $\mathcal{M}^{t,k}$ be the set of lattices $\{L_i\}_{i\in\mathbf{Z}}$ with $L_i := M_{ik+t}$. It is clear that if $M_j \in \mathcal{M}^{t,k}$, then $M_{j+ik} \in \mathcal{M}^{t,k}$ for any i. Now $L_{i+n} = M_{(i+n)k+t} = M_{ik+t+kn} = M_{ik+t}\varpi = L_i\varpi$. Hence, $\mathcal{M}^{t,k}$ is a lattice chain of period n.

LEMMA: *Let $\mathcal{M}$ be a regular small admissible lattice chain with numerical invariant (kn, m_0) for $k > 1$. Suppose that $m_0 = 0$. Then $\mathcal{M}^{0,k}$ is regular small admissible.*

Proof: We know that $\mathcal{M}^{0,k}$ is a lattice chain of period n. It is clear that $\mathcal{M}^{0,k}$ is regular because $\mathcal{M}$ is regular. Suppose kn is odd. Then we have $M_j^* = M_{-j}$ for any $j \not\equiv 0 \mod kn$. Now n is also odd and $L_i^* = M_{ik}^* = M_{-ik} = L_{-i}$ for any $i \not\equiv 0 \mod n$. Suppose kn is even and n is even. Then we have $M_j^* = M_{-j}$ for any $j \not\equiv 0$ or $kn/2 \mod kn$. Now $L_i^* = M_{ik}^* = M_{-ik} = L_{-i}$ for any $ik \not\equiv 0$ or $kn/2 \mod kn$. The condition $ik \not\equiv 0$ or $kn/2 \mod kn$ is equivalent to the condition $i \not\equiv 0$ or $n/2 \mod n$. Suppose kn is even and n is odd. Then we have $M_j^* = M_{-j}$ for any $j \not\equiv 0$ or $kn/2 \mod kn$. Clearly we have $L_i^* = M_{ik}^* = M_{-ik} = L_{-i}$ for any $i \not\equiv 0 \mod n$. Clearly condition (v) in subsection 3.1 is also satisfied for $\mathcal{M}^{0,k}$. Hence, $\mathcal{M}^{0,k}$ is a regular small admissible lattice chain with numerical $(n, 0)$. ∎

7.5.

LEMMA: *Let $\mathcal{M}$ be a regular small admissible lattice chain with numerical invariant (kn, m_0) for $k > 1$. Suppose that $m_0 = 1$ and k is odd. Then $\mathcal{M}^{k',k}$ is a regular small admissible where $k' = (k-1)/2$.*

Proof: We know that $\mathcal{M}^{k',k}$ is a regular lattice chain of period n. Suppose kn is odd. Then we have $M_j^* = M_{-j}$ for any $j \not\equiv 0 \mod kn$. It is not difficult to check that the condition $i \not\equiv (n-1)/2 \mod n$ is equivalent to the condition $ik + k' \not\equiv (kn-1)/2 \mod kn$. Now n is also odd and $L_i^* = M_{ik+k'}^* = M_{-ik-k'-1} = M_{-(i+1)k+k'} = L_{-i-1}$ for any $i \not\equiv (n-1)/2 \mod n$. Suppose

kn is even. Then we have $M_j^* = M_{-j}$ for all j. Now n is also even and $L_i^* = M_{ik+k'}^* = M_{-ik-k'-1} = M_{-(i+1)k+k'} = L_{-i-1}$ for all i. It is not difficult to see that condition (v) in subsection 3.1 is also satisfied for $\mathcal{M}^{k',k}$. Hence, $\mathcal{M}^{k',k}$ is a regular small admissible lattice chain with numerical invariant $(n,1)$. ∎

7.6.

LEMMA: *Let $\mathcal{M}$ be a regular small admissible lattice chain with numerical invariant (kn, m_0) for $k > 1$. Suppose that either (1) $m_0 = 1$, $k = 2k'+1$, $t = k'$, or (2) $m_0 = 0$, $t = 0$. Then for any $d \geq 0$ we have*

$$G_{\mathcal{M}^{t,k},(d/n)^+} \subseteq G_{\mathcal{M},(kd/kn)^+}.$$

Proof: We already know that $\mathcal{M}^{t,k} = \{L_i\}_{i\in\mathbf{Z}}$ is a regular small admissible lattice chain of period n from Lemmas 7.4 and 7.5. Let (n, n_0) denote the numerical invariant of $\mathcal{M}^{t,k}$. Suppose that g is an element in $G_{\mathcal{M}^{t,k},(d/n)^+}$. Then $(g-1).L_i \subseteq L_{i+d+1}^\sharp$ for any integer i from (3.2.2). We know

$$L_{i+d+1}^\sharp = L_{-i-d-1-n_0}^* = M_{(-i-d-1-n_0)k+t}^* = M_{(i+d+1+n_0)k-t-m_0}^\sharp.$$

Then we have $(g-1).M_{ik+t} \subseteq M_{(i+d)k+k+n_0k-t-m_0}^\sharp$ for any integer i. First, suppose that $m_0 = 1$, $k = 2k'+1$, $t = k'$. Then $n_0 = 1$ and $(g-1).M_{ik+k'} \subseteq M_{ik+k'+dk+k}^\sharp$. If j is any integer, then there is an integer i_0 such that $i_0k + k' \leq j < (i_0+1)k + k'$. Therefore, $(g-1).M_j \subseteq (g-1).M_{i_0k+k'} \subseteq M_{i_0k+k'+dk+k}^\sharp \subseteq M_{j+dk+1}^\sharp$. Secondly, suppose that $m_0 = 0$, $t = 0$. Then $n_0 = 0$ and $(g-1).M_{ik} \subseteq M_{ik+dk+k}^\sharp$. If j is any integer, then there is an integer i_0 such that $i_0k \leq j < (i_0+1)k$. Therefore, $(g-1).M_j \subseteq (g-1).M_{i_0k} \subseteq M_{i_0k+dk+k}^\sharp \subseteq M_{j+dk+1}^\sharp$.

Because g is an element in $G_{\mathcal{M}^{t,k},(d/n)^+}$, we also know that $(g-1).L_i^\sharp \subseteq L_{i+d}$ for all i. We have

$$L_i^\sharp = L_{-i-n_0}^* = M_{(-i-n_0)k+t}^* = M_{(i+n_0)k-t-m_0}^\sharp.$$

Then we get $(g-1).M_{ik+n_0k-t-m_0}^\sharp \subseteq M_{ik+dk+t}$ for all i. First suppose that $m_0 = 1$, $k = 2k'+1$, $t = k'$. Then $n_0 = 1$ and $(g-1).M_{ik+k'}^\sharp \subseteq M_{ik+k'+dk}$ for all i. If j is any integer, then there is an integer i_0 such that $i_0k+k' \leq j < (i_0+1)k+k'$. If $j = i_0k + k'$, then clearly $(g-1).M_j^\sharp \subseteq M_{j+dk}$. If $i_0k + k' < j < (i_0+1)k + k'$, then we have

$$\begin{aligned}(g-1).M_j^\sharp \subseteq (g-1).M_{j-1} \subseteq (g-1).M_{i_0k+k'} &\subseteq M_{i_0k+k'+dk+k}^\sharp \\ &\subseteq M_{i_0k+k'+dk+k-1} \\ &\subseteq M_{j+dk}.\end{aligned}$$

Secondly, suppose that $m_0 = 0$ and $t = 0$. Then $n_0 = 0$ and $(g-1).M^\sharp_{ik} \subseteq M_{ik+dk}$. If j is any integer, then there is an integer i_0 such that $i_0k \le j < (i_0+1)k$. If $j = i_0k$, then clearly $(g-1).M^\sharp_j \subseteq M_{j+dk}$. If $i_0k < j < (i_0+1)k$, then

$$(g-1).M^\sharp_j \subseteq (g-1).M_{j-1} \subseteq (g-1).M_{i_0k} \subseteq M^\sharp_{i_0k+dk+k} \subseteq M_{i_0k+dk+k-1} \subseteq M_{j+dk}.$$

Therefore, $(g-1).M_j \subseteq M^\sharp_{j+dk+1}$ and $(g-1).M^\sharp_j \subseteq M_{j+dk}$ for any integer j for both cases. Hence, g is also an element in $G_{\mathcal{M},(kd/kn)^+}$. Thus we have proved that $G_{\mathcal{M}^{t,k},(d/n)^+} \subseteq G_{\mathcal{M},(kd/kn)^+}$. ∎

7.7.

Lemma: *Let $\mathcal{M}$ be a regular small admissible lattice chain of numerical invariant $(4n,1)$. Let $\mathcal{L}$ be the collection of lattices $\{M_i \mid i \equiv 0$ or $3 \pmod 4\}$. Then $\mathcal{L}$ is a regular small admissible lattice chain of period $2n$. And for $d \ge 0$ we have*

$$G_{\mathcal{L},(2d/2n)^+} \subseteq G_{\mathcal{M},(4d/4n)^+}.$$

Proof: Because now the period of $\mathcal{M}$ is even and $m_0 = 1$, we know that $\mathcal{M}$ is self-dual from subsection 3.1. Therefore, we have $M_i = M^\sharp_i = M^*_{-i-1}$. But we know that $i \equiv 0$ or $3 \pmod 4$ if and only if $-i-1 \equiv 0$ or $3 \pmod 4$. Define $L_i := M_{2i}$ if i is even, and $L_i := M_{2i+1}$ if i odd. Then it is clear that $\mathcal{L}$ is a regular self-dual lattice chain. Hence, $\mathcal{L}$ is regular small admissible. And it is obvious that the period of $\mathcal{L}$ is $2n$.

Now if $g \in G_{\mathcal{L},(2d/2n)^+}$, then $(g-1).L_i \subseteq L_{i+2d+1}$ for any i. Then we have

$$\begin{cases} (g-1).M_{2i} \subseteq M_{2i+4d+3}, & \text{if } i \text{ is even;} \\ (g-1).M_{2i+1} \subseteq M_{2i+4d+2}, & \text{if } i \text{ is odd.} \end{cases}$$

Then we conclude that $(g-1).M_j \subseteq M_{j+4d+1}$ for any j. Because $\mathcal{M}$ is self-dual, we conclude that g is an element in $G_{\mathcal{M},(4d/4n)^+}$. Hence, we have $G_{\mathcal{L},(2d/2n)^+} \subseteq G_{\mathcal{M},(4d/4n)^+}$. ∎

7.8. Proof of Proposition 7.1.

Proof: Let k be $\gcd(e,m)$. Write $m = kn$ and $e = kd$. Suppose that $m_0 = 0$. Let $\mathcal{L}$ be the lattice chain $\mathcal{M}^{0,k}$ given in Lemma 7.4. Then clearly $d/n = e/m$, $\gcd(d,n) = 1$ and $G_{\mathcal{L},(d/n)^+} \subseteq G_{\mathcal{M},(e/m)^+}$ by Lemma 7.6.

Next suppose that $m_0 = 1$ and k is odd. Let $\mathcal{L}$ be the lattice chain $\mathcal{M}^{k',k}$ given in Lemma 7.5 where $k' = (k-1)/2$. Then we have $d/n = e/m$, $\gcd(d,n) = 1$ and $G_{\mathcal{L},(d/n)^+} \subseteq G_{\mathcal{M},(e/m)^+}$ by Lemma 7.6.

Now we suppose that $k = 2$ and $m_0 = 1$. Then just let $\mathcal{L} = \mathcal{M}$, $d = e$. Then obviously $d/n = e/m$, $\gcd(d, n) = 2$ and $G_{\mathcal{L},(d/n)^+} \subseteq G_{\mathcal{M},(e/m)^+}$.

Finally, suppose that k is a multiple of 4 and $m_0 = 1$. Write $m = 4n'$ and $e = 4d'$. Let $\mathcal{M}'$ be the lattice chain $\mathcal{L}$ given in Lemma 7.7. Then we have $2d'/2n' = e/m$, $\gcd(2d', 2n') = \gcd(e, m)/2$ and $G_{\mathcal{M}',(d'/n')^+} \subseteq G_{\mathcal{M},(e/m)^+}$. Repeat the process; the situation can be reduced to the previous two cases by induction. The proof is complete. ∎

8. Proof of Proposition 5.3

8.1. Let $\mathcal{L} = \{L_i\}_{i \in \mathbf{Z}}$ be a regular small admissible lattice chain in $\mathcal{V}$ with numerical invariant (n, n_0). Define $\mathbf{v}_i := L_i/L_{i+1}^\sharp$ for each i and $\mathbf{v} := \bigoplus_{i=0}^{n-1} \mathbf{v}_i$. Identify $\mathbf{v}_i := L_i/L_{i+1}^\sharp$ with $\mathbf{v}_{i+n} := L_{i+n}/L_{i+n+1}^\sharp$ via the multiplication by ϖ. Then $\mathbf{v}$ is a vector space over $\mathbf{f}$ and graded by $\mathbf{Z}/n\mathbf{Z}$. The quotient $\mathfrak{g}_{\mathcal{L},d/n}/\mathfrak{g}_{\mathcal{L},(d/n)^+}$ can be regarded as a subset of endomorphisms of $\mathbf{v}$. In fact, it is not difficult to see that an element in $\mathfrak{g}_{\mathcal{L},d/n}/\mathfrak{g}_{\mathcal{L},(d/n)^+}$ is a graded endomorphism of $\mathbf{v}$ of degree $-d$, i.e., it maps $\mathbf{v}_i$ to $\mathbf{v}_{i-d}$ for each i. The following lemma is from [Yu98b].

LEMMA: *Suppose that X is an element in $\mathfrak{g}_{\mathcal{L},-d/n}$. Then the following two statements are equivalent.*

(i) *The residue class $X + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ contains a nilpotent element.*

(ii) *The residue class $X + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}$.*

8.2.

LEMMA: *Suppose that one of the following three conditions:*

(i) $\gcd(d, n) = 1$ *and* $n > 1$,

(ii) $\mathcal{L}$ *is self-dual and* $n = 1$,

(iii) $\mathcal{L}$ *is self-dual and* $\gcd(d, n) = 2$,

is satisfied. Then we have

$$\bigcap_{i+j=\mu} L_i^\sharp \oslash L_j' \cap \bigcap_{i+j=\mu} L_i \oslash L_j'^\sharp = \bigcap_{i+j=\mu} L_i \oslash L_j'$$

where $\mu := (-n - n_0 - n_0' - d)/2$.

Proof: Because $L_i \subseteq L_i^\sharp$ and $L_j' \subseteq L_j'^\sharp$, it is obvious that

$$\bigcap_{i+j=\mu} L_i \oslash L_j' \subseteq \bigcap_{i+j=\mu} L_i^\sharp \oslash L_j' \cap \bigcap_{i+j=\mu} L_i \oslash L_j'^\sharp.$$

Now we consider the opposite inclusion. If either $L_i = L_i^\sharp$ or $L'_j = L'^\sharp_j$, then it is clear that $L_i^\sharp \odot L'_j \cap L_i \odot L'^\sharp_j \subseteq L_i \odot L'_j$. Hence, the opposite inclusion is obvious when $\mathcal{L}$ or $\mathcal{L}'$ is self-dual. So now we only need to consider the case when $\gcd(d,n) = 1$. Because we assume that $n + n_0 + n'_0 + d$ is even and $\gcd(d,n) = 1$, $n_0 + n'_0$ has to be odd when n is even. Suppose that n is even and $n_0 = 1$. In this case we know that $\mathcal{L}$ is self-dual. Suppose that n is even, $n_0 = 0$ and $n'_0 = 1$. In this case we know that $\mathcal{L}'$ is self-dual.

Suppose that n is odd, $n_0 = 0$ and $n'_0 = 0$. If $L_i \neq L_i^\sharp$ and $L'_j \neq L'^\sharp_j$ for some i, j such that $i + j = \mu$, then i and j are multiples of n from subsection 3.1. Hence, $\mu := (-n - d - n_0 - n'_0)/2$ is a multiple of n. So d is a multiple of n. This contradicts the assumption that $\gcd(d,n) = 1$ and $n > 1$.

Suppose that n is odd, $n_0 = 0$ and $n'_0 = 1$. If $L_i \neq L_i^\sharp$ and $L'_j \neq L'^\sharp_j$ for some i, j such that $i + j = \mu$, then i is a multiple of n and $j \equiv (n-1)/2 \pmod{n}$. Hence, we get $(-n - d - n_0 - n'_0)/2 \equiv (n-1)/2 \pmod{n}$. So d is a multiple of n. This contradicts the assumption that $\gcd(d,n) = 1$ and $n > 1$. Suppose that n is odd, $n_0 = 1$ and $n'_0 = 0$. The situation is the same as the previous case.

Suppose that n is odd, $n_0 = 1$ and $n'_0 = 1$. If $L_i \neq L_i^\sharp$ and $L'_j \neq L'^\sharp_j$ for some i, j such that $i + j = \mu$, then $i \equiv (n-1)/2 \pmod{n}$ and $j \equiv (n-1)/2 \pmod{n}$. Hence, we get $(-n - d - 2)/2 \equiv n - 1 \pmod{n}$. So d is a multiple of n. This contradicts the assumption that $\gcd(d,n) = 1$ and $n > 1$.

So we have proved that if $\gcd(d,n) = 1$, then there is no i such that $L_i \neq L_i^\sharp$ and $L'_{\mu-i} \neq L'^\sharp_{\mu-i}$. So we have either $L_i = L_i^\sharp$ or $L'_{\mu-i} = L'^\sharp_{\mu-i}$ for all i. Hence,

$$\bigcap_{i+j=\mu} L_i^\sharp \odot L'_j \cap \bigcap_{i+j=\mu} L_i \odot L'^\sharp_j \subseteq \bigcap_{i+j=\mu} L_i \odot L'_j.$$

The proof is complete. ∎

8.3.

Lemma: *Let w be an element in $B(\mathcal{L}, \mathcal{L}', d/n)$. Then $\mathfrak{M}(w) + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}$ if and only if $\mathfrak{M}'(w) + \mathfrak{g}'_{\mathcal{L}',(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}'$.*

Proof: It is not difficult to see that $\mathfrak{M}(w) + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}$ if and only if there exists a number k such that $\mathfrak{M}(w)^k(L_i) \subseteq L_{i-kd}\varpi$ for all i. Similarly, $\mathfrak{M}'(w) + \mathfrak{g}'_{\mathcal{L}',(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}'$ if and only if there exists a number k such that $\mathfrak{M}'(w)^k(L'_i) \subseteq L'_{i-kd}\varpi$ for all i. Write

$w = \sum_i a_i \otimes b_i$ for $a_i \in \mathcal{V}$ and $b_i \in \mathcal{V}'$. We know that

$$\mathfrak{M}(w) = \frac{1}{2}\varpi_F^{1-\lambda_F} \sum_{i,j} c_{a_i,a_j\langle b_j,b_i\rangle'} \quad \text{and} \quad \mathfrak{M}'(w) = \frac{1}{2}\varpi_F^{1-\lambda_F} \sum_{i,j} c_{b_i,b_j\langle a_j,a_i\rangle}$$

from subsection 4.3. Note that $c_{a,b}(x) = a\langle b,x\rangle - \epsilon b\langle a,x\rangle$. Hence, it is clear that $\mathfrak{M}(w) + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}$ if and only if $\mathfrak{M}'(w) + \mathfrak{g}'_{\mathcal{L}',(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}'$. ∎

8.4. Proof of (i) of Proposition 5.3.

Proof: The endomorphism $(\mathfrak{M}'(w) + \mathfrak{g}'_{\mathcal{L}',(-d/n)^+})^n$ is a combination of compositions

$$\mathbf{v}'_i \to \mathbf{v}'_{i-d} \to \cdots \to \mathbf{v}'_{i-(n-1)d} \to \mathbf{v}'_{i-nd} = \mathbf{v}'_i \tag{8.4.1}$$

for all $i \in \mathbf{Z}/n\mathbf{Z}$. Now suppose that there are two lattices in $\mathcal{L}'$ which coincide. Hence, a certain $\mathbf{v}'_{i_0}$ is trivial. From the assumption $\gcd(d,n) = 1$ there is a number k such that $0 \le k < n$ and $i - kd \equiv i_0 \pmod n$ for any i. Hence, $(\mathfrak{M}'(w) + \mathfrak{g}'_{\mathcal{L}',(-d/n)^+})^n$ is trivial. Therefore, $\mathfrak{M}'(w) + \mathfrak{g}'_{\mathcal{L}',(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}'$. Then by Lemma 8.3, $\mathfrak{M}(w) + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ is a nilpotent endomorphism of $\mathbf{v}$. Therefore, by Lemma 8.1 the residue class $\mathfrak{M}(w) + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ contains a nilpotent element. But we know that the coset $\mathfrak{M}(w)+\mathfrak{g}_{\mathcal{L},(-d/n)^+}$ presents the character ψ_w from subsections 5.4 and 5.5. Hence, the coset $\mathfrak{M}(w) + \mathfrak{g}_{\mathcal{L},(-d/n)^+}$ cannot contain any nilpotent element by the argument before Proposition 5.3. Thus we obtain a contradiction. Therefore, any two lattices in $\mathcal{L}'$ must be distinct. ∎

8.5. Now we start the preparation for the proof of (ii) of Proposition 5.3. First, we have to construct the lattice chains $\mathcal{M}$ and $\mathcal{M}'$. The construction is in this and the next subsections. The main body of the proof will be in subsection 8.10.

For simplicity in notation we will replace n, d by $2n, 2d$ respectively from now on. So now we have $\gcd(d,n) = 1$. Moreover, we know that $\mathcal{L} = \{L_i\}_{i\in\mathbf{Z}}$ is a regular small admissible lattice chain in $\mathcal{V}$ with numerical invariant $(2n,1)$, $\mathcal{L}' = \{L'_i\}_{i\in\mathbf{Z}}$ is a non-regular small admissible lattice chain in $\mathcal{V}'$ with numerical invariant $(2n,1)$, and w is an element in $B(\mathcal{L},\mathcal{L}',2d/2n)$. We note that both $\mathcal{L}$ and $\mathcal{L}'$ are self-dual in the present situation. Because $\mathcal{L}'$ is not regular, we assume that there is an l such that $0 \le l < n$ and $L'_l = L'_{l+1}$. Define a number

$$\eta' := \begin{cases} -d-1, & \text{if } l+d \text{ is odd;} \\ -n-d-1, & \text{if } l+d \text{ is even and } n \text{ is odd;} \\ -2d-1, & \text{if } l+d \text{ is even and } n \text{ is even.} \end{cases} \tag{8.5.1}$$

LEMMA: *There is a unique integer k' such that $L'_{\eta'-2k'd}$ is similar to L'_l or L'^*_{l+1} where $0 \leq k' \leq \lfloor \frac{n-1}{2} \rfloor$ for the first two cases in (8.5.1) and $0 \leq k' \leq \frac{n}{2} - 1$ for the third case.*

Proof: We know that $L'^*_{l+1} = L'_{-l-2}$ from subsection 3.1 because now $n'_0 = 1$. Here we need to prove that there is a number k' such that $\eta' - 2k'd - l$ or $\eta' - 2k'd + l + 2$ is a multiple of $2n$. Note that both $\eta' - l$ and $\eta' + l + 2$ are even for all three cases. Because now $\gcd(d, n) = 1$, there exist integers i_1, i_2, j_1, j_2 such that $\eta' - l = 2i_1 d + 2j_1 n$ and $\eta' + l + 2 = 2i_2 d + 2j_2 n$. We may assume that $0 \leq i_1, i_2 < n$. Now

$$\begin{aligned} 2j_1 n + 2j_2 n &= \eta' - l - 2i_1 d + \eta' + l + 2 - 2i_2 d \\ &= \begin{cases} -2(i_1 + i_2 + 1)d, & \text{if } l + d \text{ is odd;} \\ -2(i_1 + i_2 + 1)d - 2n, & \text{if } l + d \text{ is even and } n \text{ is odd;} \\ -2(i_1 + i_2 + 2)d, & \text{if } l + d \text{ is even and } n \text{ is even.} \end{cases} \end{aligned}$$

This implies that the number $i_1 + i_2 + 1$ divides n for the first two cases. By our assumption, we conclude that $i_1 + i_2 + 1 = n$ for the first two cases. Therefore, one of i_1, i_2 is less than or equal to $\lfloor \frac{n-1}{2} \rfloor$. For the last case $i_1 + i_2 + 2$ divides n. Then $i_1 + i_2 + 2 = n$. Therefore, one of i_1, i_2 is less than or equal to $\frac{n}{2} - 1$. The uniqueness of k' is obvious. Thus the lemma is proved. ∎

Let k' be the number given by the above lemma. Define a set $\mathcal{M}'_0$ of lattices to be

(8.5.2)

$$\begin{cases} \{L'_{\eta'-2id}\}_{0 \leq i \leq k'} \cup \{L'_{\eta'+1-2id}\}_{k'+1 \leq i < n/2-1} \cup \{\Gamma', \Gamma'^*_{\mathrm{m}}\}, & \text{if } n \text{ even, } l \text{ odd;} \\ \{L'_{\eta'-2id}\}_{0 \leq i \leq k'} \cup \{L'_{\eta'+1-2id}\}_{k'+1 \leq i < n/2}, & \text{if } n \text{ even, } l \text{ even;} \\ \{L'_{\eta'-2id}\}_{0 \leq i \leq k'} \cup \{L'_{\eta'+1-2id}\}_{k'+1 \leq i < (n-1)/2} \cup \{\Gamma'\}, & \text{if } n \text{ odd, } l \text{ odd;} \\ \{L'_{\eta'-2id}\}_{0 \leq i \leq k'} \cup \{L'_{\eta'+1-2id}\}_{k'+1 \leq i < (n-1)/2} \cup \{\Gamma'^*_{\mathrm{m}}\}, & \text{if } n \text{ odd, } l \text{ even} \end{cases}$$

where Γ' (resp. Γ'_{m}) is a fixed maximal (resp. minimal) good lattice in $\mathcal{V}'$ such that $L' \subseteq \Gamma'$ (resp. $\Gamma'_{\mathrm{m}} \subseteq L'$) for any good lattice L' in $\mathcal{L}'$. Now the number of elements in $\mathcal{M}'_0$ is $\frac{n}{2} + 1$ for the first case and is $\lfloor \frac{n+1}{2} \rfloor$ for others. Therefore, $\mathcal{M}'_0$ can generate a small admissible lattice chain of period n as follows. We know that a lattice $L' \in \mathcal{L}'$ is similar to a good lattice $\widetilde{L}'$ or the dual lattice of a good lattice $\widetilde{L}'$. Define

$$\mathcal{M}'_1 := \{\widetilde{L}', \widetilde{L}'^* \mid L' \in \mathcal{M}'_0 - \{\Gamma', \Gamma'^*_{\mathrm{m}}\}\} \cup (\{\Gamma', \Gamma'^*_{\mathrm{m}}\} \cap \mathcal{M}'_0).$$

Now $\mathcal{M}'_1$ is a set of n lattices in $\mathcal{V}'$. Denote lattices in $\mathcal{M}'_1$ by

$$M'_{-\lfloor (n+1)/2 \rfloor}, M'_{-\lfloor (n+1)/2 \rfloor + 1}, \ldots, M'_{\lfloor (n-1)/2 \rfloor}$$

such that $M'_i \subseteq M'_j$ if $i \geq j$. Let $\mathcal{M}' = \{M'_i\}_{i\in\mathbf{Z}}$ be the lattice chain generated by $\mathcal{M}'_1$ via the formula $M'_{i+n} = M'_i\varpi$. It is clear that $\mathcal{M}'$ is a small admissible of period n.

8.6. Next we can define a small admissible lattice chain $\mathcal{M}$ in $\mathcal{V}$ similar to $\mathcal{M}'$. Define

$$\eta := \begin{cases} -d-1, & \text{if } l+n \text{ is odd;} \\ -n-d-1, & \text{if } l \text{ is odd and } n \text{ is odd;} \\ -2d-1, & \text{if } l \text{ is even and } n \text{ is even.} \end{cases} \tag{8.6.1}$$

Similar to Lemma 8.5, we have the following lemma.

LEMMA: *There is a unique integer k such that $L'_{-n-1-d-\eta-2kd}$ is similar to L'^*_l or L'_{l+1} where $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ for the first two cases in (8.6.1) and $0 \leq k \leq n/2$ otherwise.*

Proof: Now we have that $L'^*_l = L'_{-l-1}$. Note that both $-n-1-d-\eta-l-1$ and $-n-1-d-\eta+l+1$ are even for the three cases. Because now $\gcd(d,n) = 1$, there exist integers i_3, i_4, j_3, j_4 such that $-n-1-d-\eta-l-1 = 2i_3d + 2j_3n$ and $-n-1-d-\eta+l+1 = 2i_4d + 2j_4n$. We assume that $0 \leq i_3, i_4 < n$. Now

$$\begin{aligned} 2j_3n + 2j_4n &= -n-1-d-\eta-l-1-2i_3d-n-1-d-\eta+l+1-2i_4d \\ &= \begin{cases} -2(i_3+i_4)d-2n, & \text{if } l+n \text{ is odd;} \\ -2(i_3+i_4)d-4n, & \text{if } l \text{ is odd and } n \text{ is odd;} \\ -2(i_3+i_4-1)d-2n, & \text{if } l \text{ is even and } n \text{ is even.} \end{cases} \end{aligned}$$

This implies that the number $i_3 + i_4$ divides n for the first two cases. By our assumption, we conclude that $i_3 + i_4 = n$ for the first two cases. Therefore, one of i_3, i_4 is less than or equal to $\lfloor \frac{n}{2} \rfloor$. For the last case $i_3 + i_4 - 1$ divides n. Hence, $i_3 + i_4 - 1 = n$. Therefore, one of i_3, i_4 is less than or equal to $n/2$. The uniqueness of k is again obvious. ∎

Similarly, let $\mathcal{M}_0$ be

(8.6.2)

$$\begin{cases} \{L_{\eta-2id}\}_{0\leq i\leq k-1} \cup \{L_{\eta+1-2id}\}_{k\leq i<n/2-1} \cup \{\Gamma, \Gamma^*_{\mathrm{m}}\}, & \text{if } n \text{ even, } l+d \text{ odd;} \\ \{L_{\eta-2id}\}_{0\leq i\leq k-1} \cup \{L_{\eta+1-2id}\}_{k\leq i<n/2}, & \text{if } n \text{ even, } l+d \text{ even;} \\ \{L_{\eta-2id}\}_{0\leq i\leq k-1} \cup \{L_{\eta+1-2id}\}_{k\leq i<(n-1)/2} \cup \{\Gamma\}, & \text{if } n \text{ odd, } l+d \text{ even;} \\ \{L_{\eta-2id}\}_{0\leq i\leq k-1} \cup \{L_{\eta+1-2id}\}_{k\leq i<(n-1)/2} \cup \{\Gamma^*_{\mathrm{m}}\}, & \text{if } n \text{ odd, } l+d \text{ odd,} \end{cases}$$

where Γ (resp. Γ_{m}) is a fixed maximal (resp. minimal) good lattice in $\mathcal{V}$ such that $L \subseteq \Gamma$ (resp. $\Gamma_{\mathrm{m}} \subseteq L$) for any good lattice L in $\mathcal{L}$. Let $\mathcal{M} = \{M_i\}_{i\in\mathbf{Z}}$ be the small admissible lattice chain of period n generated by $\mathcal{M}_0$ as described in subsection 8.5.

8.7.

LEMMA: *Let k' and k be the integers given by Lemma 8.5 and 8.6, respectively. Then*

$$k+k' = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd;} \\ \frac{n}{2}-1 & \text{if } l \text{ is odd and } n \text{ is even;} \\ \frac{n}{2} & \text{if } l \text{ is even and } n \text{ is even.} \end{cases}$$

Proof: Suppose that l is even, d is odd and n is odd. So we have $\eta' = -d-1$, $\eta = -d-1$ from (8.5.1) and (8.6.1). Clearly, if $L'_{\eta'-2k'd}$ is similar to L'_l, then the lattice $L'_{-n-1-d-\eta-2kd}$ is similar to L'^*_l; if $L'_{\eta'-2k'd}$ is similar to L'^*_{l+1}, then $L'_{-n-1-d-\eta-2kd}$ is similar to L'_{l+1}. So we have

$$(\eta' - 2k'd) + (-n-1-d-\eta-2kd) \equiv -1 \pmod{n}.$$

Therefore, $(2k'+2k+1)d$ is a multiple of n. Because $\gcd(d,n)=1$, we know that $2k'+2k+1$ is a multiple of n. Because $0 \le k \le \lfloor \frac{n}{2} \rfloor$ and $0 \le k' \le \lfloor \frac{n-1}{2} \rfloor$, we have $2k'+2k+1 = n$. Hence, $k+k' = (n-1)/2$. The proofs for other cases are similar, so we omit them. ∎

8.8.

LEMMA: *Let $\mathcal{M}, \mathcal{M}'$ be the admissible lattice chains of period n defined as above. Let (n, m_0) (resp. (n, m'_0)) be the numerical invariant of $\mathcal{M}$ (resp. $\mathcal{M}'$). Then $n + m_0 + m'_0 + d$ is even.*

Proof: From the construction of the lattice chains $\mathcal{M}, \mathcal{M}'$ it is clear that

$$(m_0, m'_0) = \begin{cases} (1,1), & \text{if } n+d \text{ is even and } l \text{ is even;} \\ (0,0), & \text{if } n+d \text{ is even and } l \text{ is odd;} \\ (0,1), & \text{if } n+d \text{ is odd and } l \text{ is even;} \\ (1,0), & \text{if } n+d \text{ is odd and } l \text{ is odd.} \end{cases} \tag{8.8.1}$$

Therefore, $n + m_0 + m'_0 + d$ is even for all four cases. ∎

8.9. AN EXAMPLE. Suppose that $2n = 14$, $2d = 6$, $L'_0 = L'_1$, i.e., $l = 0$. Now we have $\eta' = -4$, $L'_{-4} = L'^*_3$, $L'_{-4-6} = L'_{-10} = L'_4\varpi^{-1}$, $L'_{-4-12} = L'_{-16} = L'^*_1\varpi^{-1}$. Therefore $k' = 2$,

$$\mathcal{M}'_0 = \{\Gamma'^*_{\mathrm{m}}, L'^*_3, L'_4\varpi^{-1}, L'^*_1\varpi^{-1}\},$$
$$\mathcal{M}'_1 = \{\Gamma'^*_{\mathrm{m}}, L'^*_4, L'^*_3, L'^*_1, L'_1, L'_3, L'_4\},$$

and $\mathcal{M}$ is the small admissible lattice chain

(8.9.1)

$$\cdots \subsetneq L'_1\varpi \subsetneq L'^*_1\varpi \subsetneq L'^*_3\varpi \subsetneq L'^*_4\varpi \subsetneq \Gamma'^*_{\mathrm{m}}\varpi \subsetneq L'_4 \subsetneq L'_3 \subsetneq L'_1 \subsetneq L'^*_1 \subsetneq \cdots$$

with numerical invariant $(7,1)$. Also, we have $\eta=-4$, $L'_{-7-1-3-(-4)+6}=L'_{-1}=L_0'^*$. Then $k=1$ and $L_{-4}=L_3^*$, $L_{-4+1-6}=L_{-9}=L_5\varpi^{-1}$, $L_{-4+1-12}=L_{-15}=L_0^*\varpi^{-1}$. Hence, $\mathcal{M}_0=\{\Gamma_{\rm m}^*, L_3^*, L_5\varpi^{-1}, L_0^*\varpi^{-1}\}$ and $\mathcal{M}$ is the small admissible lattice chain

$$(8.9.2)\quad \cdots\subsetneq L_0\varpi\subsetneq L_0^*\varpi\subsetneq L_3^*\varpi\subsetneq L_5^*\varpi\subsetneq \Gamma_{\rm m}^*\varpi\subsetneq L_5\subsetneq L_3\subsetneq L_0\subsetneq L_0^*\subsetneq\cdots$$

with numerical invariant $(7,1)$. The lattice chains $\mathcal{M}=\{M_i\}_{i\in\mathbf{Z}}$, $\mathcal{M}'=\{M_i'\}_{i\in\mathbf{Z}}$ are indexed such that $M_0=L_0$ and $M_0'=L_1'$. So $M_{-8}=L_0^*\varpi^{-1}$, $M_{-7}=L_0\varpi^{-1}$, $M_{-6}=L_3\varpi^{-1}$, $M_{-5}=L_5\varpi^{-1}$, $M_{-4}=\Gamma_{\rm m}^*$, $M_{-3}=L_5^*$, $M_{-2}=L_3^*$, $M_2'=L_4'$, $M_1'=L_3'$, $M_0'=L_1'$, $M'_{-1}=L_1'^*$, $M_{-2}=L_3'^*$, $M_{-3}=L_4'^*$, $M_{-4}=\Gamma_{\rm m}'^*$. By Lemma 8.2 we have $B(\mathcal{L},\mathcal{L}',\frac{6}{14})=\bigcap_{i+j=-11}L_i\oslash L_j'=\bigcap_{i=-17}^{-4}L_i\oslash L'_{-11-i}$, and

$$\begin{aligned}B\Big(\mathcal{M},\mathcal{M}',\frac{3}{7}\Big)=&\bigcap_{i=-8}^{-2}M_i\oslash M'_{-6-i}\\
=&L_0^*\varpi^{-1}\oslash L_4'\cap L_0\varpi^{-1}\oslash L_3'\cap L_3\varpi^{-1}\oslash L_1'\\
&\qquad\cap L_5\varpi^{-1}\oslash L_1'^*\cap\Gamma_{\rm m}^*\oslash L_3'^*\cap L_5^*\oslash L_4'^*\cap L_3^*\oslash\Gamma_{\rm m}'^*\\
=&L_{-15}\oslash L_4'\cap L_{-14}\oslash L_3'\cap L_{-11}\oslash L_1'\cap L_{-9}\oslash L'_{-2}\\
&\qquad\cap\Gamma_{\rm m}^*\oslash L'_{-4}\cap L_{-6}\oslash L'_{-5}\cap L_{-4}\oslash\Gamma_{\rm m}'^*.\end{aligned}$$

Because $L_1'=L_0'$, $L'_{-7}=L_6'^*\subsetneq\Gamma_{\rm m}'^*$ and $L_{-7}=L_6^*\subsetneq\Gamma_{\rm m}^*$, it is clear that $B(\mathcal{L},\mathcal{L}',\frac{6}{14})\subseteq B(\mathcal{M},\mathcal{M}',\frac{3}{7})$.

8.10. Proof of (ii) of Proposition 5.3.

Proof: Recall that d,n have been replaced by $2d,2n$ since subsection 8.5. So now we have $\gcd(d,n)=1$. We know that the numerical invariant of $\mathcal{L}$ and $\mathcal{L}'$ are both $(2n,1)$. Then $(-2n-2d-n_0-n_0')/2=-n-d-1$. So we have

$$\begin{aligned}B(\mathcal{L},\mathcal{L}',2d/2n)&=\bigcap_{i+j=-n-d-1}L_i\oslash L_j',\\
B(\mathcal{M},\mathcal{M}',d/n)&=\bigcap_{i+j=(-n-m_0-m_0'-d)/2}M_i\oslash M_j',\end{aligned}$$

by Lemma 8.2. Recall that Γ (resp. $\Gamma_{\rm m}$) is a fixed maximal (resp. minimal) good lattice in $\mathcal{V}$ such that $L\subseteq\Gamma$ (resp. $\Gamma_{\rm m}\subseteq L$) for any good lattice L in $\mathcal{L}$, and Γ' (resp. $\Gamma'_{\rm m}$) is a fixed maximal (resp. minimal) good lattice in $\mathcal{V}'$ such that $L'\subseteq\Gamma'$ (resp. $\Gamma'_{\rm m}\subseteq L'$) for any good lattice L' in $\mathcal{L}'$. We shall prove the proposition by discussing the six cases according to the parity of n,l,d. There are only six cases because n,d cannot both be even. For each case we define a lattice C in $\mathcal{W}$.

Then we check that $C = B(\mathcal{M}, \mathcal{M}', d/n)$ and $B(\mathcal{L}, \mathcal{L}', 2d/2n) \subseteq C$. The proof will be written in detail for the first case and be sketchy for the others because the checking is elementary and analogous to the first case.

(1) Suppose that l is even, d is odd, n is odd. Then we have $\eta = \eta' = -d-1$, $m_0 = m_0' = 1$ from (8.5.1), (8.6.1) and Lemma 8.8. Define the lattice C to be (8.10.1)

$$\begin{aligned}
&\Gamma_{\mathrm{m}}^* \odot L'_{\eta'} \cap \bigcap_{1\le i\le k'} (L^*_{-n+2id} \odot L'^*_{\eta'-2(i-1)d}\varpi^{-1} \cap L_{-n+2id} \odot L'_{\eta'-2id})\\
&\cap L^*_{-n-1+2(k'+1)d} \odot L'^*_{\eta'-2k'd}\varpi^{-1} \cap L_{-n-1+2(k'+1)d} \odot L'_{\eta'+1-2(k'+1)d}\\
&\cap \bigcap_{k'+1<i<(n-1)/2} (L^*_{-n-1+2id} \odot L'^*_{\eta'+1-2(i-1)d}\varpi^{-1} \cap L_{-n-1+2id} \odot L'_{\eta'+1-2id})\\
&\cap L^*_{-n-1+(n-1)d} \odot L'^*_{\eta'+1-(n-3)d}\varpi^{-1} \cap L_{-n-1+(n-1)d} \odot \Gamma'^*_{\mathrm{m}}\varpi^{-(d-1)/2}.
\end{aligned}$$

First, the lattices L'_j which appear in (8.10.1) clearly belong to $\mathcal{M}'$ from (8.5.2), so does Γ'^*_{m}. Secondly, we have $L_{-n-1+(n-1)d} \sim L_{-d-1} = L_\eta$ and $L_{-n-1+2id} \sim L_{\eta-2((n-1)/2-i)d}$. We know that $k+k' = (n-1)/2$ from Lemma 8.7. Therefore, Γ_{m} and the lattices L_j which appear in (8.10.1) belong to $\mathcal{M}$ from (8.6.2). Thirdly, it is not difficult to check that $\Gamma^*_{\mathrm{m}} = M_{-(n+1)/2}$ and $L'_{\eta'} = M'_{\eta'/2} = M'_{(-d-1)/2}$. Therefore, we have

$$\begin{aligned}
C &= \bigcap_{i+j=(-n-1)/2+(-d-1)/2} M_i \odot M'_j\\
&= \bigcap_{i+j=(-n-m_0-m_0'-d)/2} M_i \odot M'_j = B(\mathcal{M}, \mathcal{M}', d/n)
\end{aligned}$$

by Lemma 8.2. Finally, every term $L_i \odot L'_j$ which appears in (8.10.1) clearly satisfies the condition that $i+j = -n-d-1$ except for the term $L^*_{-n-1+2(k'+1)d} \odot L'^*_{\eta'-2k'd}\varpi^{-1}$. Now we have $L^*_{-n-1+2(k'+1)d} = L_{n+1-2(k'+1)d-1}$, $L'^*_{\eta'-2k'd}\varpi^{-1} = L'_{-\eta'+2k'd-1-2n}$ and $n+1-2(k'+1)d-1-\eta'+2k'd-1-2n = -n-d$. We also have $L'^*_{\eta'-2k'd} = L'^*_{\eta'-2k'd+1}$ because $L'^*_{\eta'-2k'd}$ is similar to L'_l or L'^*_{l+1} and $L'_l = L'_{l+1}$. So $B(\mathcal{L}, \mathcal{L}', 2d/2n) \subseteq L^*_{-n-1+2(k'+1)d} \odot L'^*_{\eta'-2k'd}\varpi^{-1}$. Hence, we conclude $B(\mathcal{L}, \mathcal{L}', 2d/2n) \subseteq C$.

(2) Suppose that l is even, d is odd, n is even. Then we have $\eta = -2d-1$, $\eta' = -d-1$ from (8.5.1) and (8.6.1). We also have $m_0 = 0$ and $m_0' = 1$ from Lemma 8.8. Moreover, it is not difficult to check that $L'_{\eta'} = M'_{\eta'/2}$. Define the

lattice C to be

$$\begin{aligned}
&\Gamma^*_{\mathrm{m}} \otimes L'_{\eta'} \cap \bigcap_{1\le i\le k'} (L^*_{-n+2id} \otimes L'^*_{\eta'-2(i-1)d}\varpi^{-1} \cap L_{-n+2id} \otimes L'_{\eta'-2id})\\
&\cap L^*_{-n-1+2(k'+1)d} \otimes L'^*_{\eta'-2k'd}\varpi^{-1} \cap L_{-n-1+2(k'+1)d} \otimes L'_{\eta'+1-2(k'+1)d}\\
&\cap \bigcap_{k'+1<i<n/2} (L^*_{-n-1+2id} \otimes L'^*_{\eta'+1-2(i-1)d}\varpi^{-1} \cap L_{-n-1+2id} \otimes L'_{\eta'+1-2id})\\
&\cap \Gamma\varpi^{-(d+1)/2} \otimes L'^*_{\eta'+1-2(n/2-1)d}.
\end{aligned}$$

We just note that $L_{-n-1+2(n/2-1)d} \sim L_{-2d-1} = L_\eta$. Then everything is completely analogous to case (1).

(3) Suppose that l is even, d is even, n is odd. Then we have $\eta = -d-1$, $\eta' = -n-d-1$, $m_0 = 0$, $m'_0 = 1$, and $L'_{\eta'} = M'_{\eta'/2}$. Define the lattice C to be

$$\begin{aligned}
&\Gamma \otimes L'_{\eta'} \cap \bigcap_{1\le i\le k'} (L^*_{2id} \otimes L'^*_{\eta'-2(i-1)d}\varpi^{-1} \cap L_{2id} \otimes L'_{\eta'-2id})\\
&\cap L^*_{-1+2(k'+1)d} \otimes L'^*_{\eta'-2k'd}\varpi^{-1} \cap L_{-1+2(k'+1)d} \otimes L'_{\eta'+1-2(k'+1)d}\\
&\cap \bigcap_{k'+1<i<(n-1)/2} (L^*_{-1+2id} \otimes L'^*_{\eta'+1-2(i-1)d}\varpi^{-1} \cap L_{-1+2id} \otimes L'_{\eta'+1-2id})\\
&\cap L^*_{-1+(n-1)d} \otimes L'^*_{\eta'+1-(n-3)d}\varpi^{-1} \cap L_{-1+(n-1)d} \otimes \Gamma'^*_{\mathrm{m}}\varpi^{-d/2}.
\end{aligned}$$

We note that $L_{-1+(n-1)d} \sim L_{-1-d} = L_\eta$. Also, we have $L^*_{2id} = L_{-2id-1}$ and $L'^*_{\eta'-2(i-1)d}\varpi^{-1} = L'_{-\eta'+2(i-1)d-1-2n} = L'_{-n-d+2id}$.

(4) Suppose that l is odd, d is odd, n is odd. Then $\eta = \eta' = -n-d-1$, $m_0 = m'_0 = 0$, $L'_{\eta'} = M'_{(\eta'+1)/2}$. Define C to be

$$\begin{aligned}
&\Gamma \otimes L'_{\eta'} \cap \bigcap_{1\le i\le k'} (L^*_{2id} \otimes L'^*_{\eta'-2(i-1)d}\varpi^{-1} \cap L_{2id} \otimes L'_{\eta'-2id})\\
&\cap L^*_{-1+2(k'+1)d} \otimes L'^*_{\eta'-2k'd}\varpi^{-1} \cap L_{-1+2(k'+1)d} \otimes L'_{\eta'+1-2(k'+1)d}\varpi^{-1}\\
&\cap \bigcap_{k'+1<i<(n-1)/2} (L^*_{-1+2id} \otimes L'^*_{\eta'+1-2(i-1)d}\varpi^{-1} \cap L_{-1+2id} \otimes L'_{\eta'+1-2id})\\
&\cap L^*_{-1+(n-1)d} \otimes L'^*_{\eta'+1-(n-3)d}\varpi^{-1} \cap L_{-1+(n-1)d} \otimes \Gamma'\varpi^{-(d+1)/2}.
\end{aligned}$$

This is analogous to case (3).

(5) Suppose that l is odd, d is odd, n is even. Then $\eta = -d-1$, $\eta' = -2d-1$,

$m_0 = 1$, $m'_0 = 0$, $L'_{\eta'} = M'_{(\eta'+1)/2}$. Define C to be

$$
\begin{aligned}
&L_{-n-d-1} \odot \Gamma' \\
&\cap \bigcap_{0\le i\le k'} (L^*_{-n-d-1-2id}\varpi^{-1} \odot L'_{\eta'-2id} \cap L_{-n-d-1-2(i+1)d} \odot L'^*_{\eta'-2id}) \\
&\cap L^*_{-n-d-1-2(k'+1)d}\varpi^{-1} \odot L'_{\eta'+1-2(k'+1)d} \cap L_{-n-d-2(k'+2)d} \odot L'^*_{\eta'+1-2(k'+1)d} \\
&\cap \bigcap_{k'+1<i<n/2-1} (L^*_{-n-d-2id}\varpi^{-1} \odot L'_{\eta'+1-2id} \cap L_{-n-d-2(i+1)d} \odot L'^*_{\eta'+1-2id}) \\
&\cap L^*_{-n-d-2(n/2-1)d} \odot \Gamma'^*_{\mathrm{m}}\varpi^{-(d+1)/2}.
\end{aligned}
$$

We note that $L^*_{-n-d-2(n/2-1)d} = L_{n+d+2(n/2-1)d-1} \sim L_{-d-1} = L_\eta$. We also have $L^*_{-n-d-1-2id}\varpi^{-1} = L_{-n+d+2id}$, $L'_{\eta'-2id} = L'_{-2d-1-2id}$ and

$$L'^*_{\eta'-2id} = L'^*_{-2d-1-2id} = L'_{2(i+1)d}.$$

(6) Suppose that l is odd, d is even, n is odd. Then $\eta = -n-d-1$, $\eta' = -d-1$, $m_0 = 1$, $m'_0 = 0$, $L'_{\eta'} = M'_{(\eta'+1)/2}$. Define C to be

$$
\begin{aligned}
&\Gamma^*_{\mathrm{m}} \odot L'_{\eta'} \cap \bigcap_{1\le i\le k'} (L^*_{-n+2id} \odot L'^*_{\eta'-2(i-1)d}\varpi^{-1} \cap L_{-n+2id} \odot L'_{\eta'-2id}) \\
&\cap L^*_{-n-1+2(k'+1)d} \odot L'^*_{\eta'-2k'd}\varpi^{-1} \cap L_{-n-1+2(k'+1)d} \odot L'_{\eta'+1-2(k'+1)d} \\
&\cap \bigcap_{k'+1<i<(n-1)/2} (L^*_{-n-1+2id} \odot L'^*_{\eta'+1-2(i-1)d}\varpi^{-1} \cap L_{-n-1+2id} \odot L'_{\eta'+1-2id}) \\
&\cap L^*_{-n-1+(n-1)d} \odot L'^*_{\eta'+1-(n-3)d}\varpi^{-1} \cap L_{-n-1+(n-1)d} \odot \Gamma'\varpi^{-d/2}.
\end{aligned}
$$

Here we just need to notice that $L_{-n-1+(n-1)d} \sim L_{-n-d-1} = L_\eta$.

So we have proved that $B(\mathcal{L}, \mathcal{L}, 2d/2n) \subseteq B(\mathcal{M}, \mathcal{M}', d/n)$. Hence, w is an element in $B(\mathcal{M}, \mathcal{M}', d/n)$. It is clear that $\mathcal{M}$ is regular because $\mathcal{L}$ is regular. Now $\gcd(d, n) = 1$ (recall that d, n is replaced by $2d, 2n$ in the proof), so $\mathcal{M}'$ must be regular by (i) of the proposition. ∎

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